Loop Quantum Gravity

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Plan

Introduction...

What do we expect from a Quantum Theory Gravity? *Minimally,*

"Explain" Black Hole Entropy;

"Resolve" the cosmological (and black hole) singularities;

Provide a "consistent" perturbation theory of gravitons (?!) .

Classical General Relativity

General Covariance: Most explicitly manifested in the Hamiltonian formulation by the constraint algebra generating action of active diffeomorphisms on the phase space.

Canonical fields: Metric q_{ab} and π^{ab} $(:= \sqrt{q}(K^{ab} - Kq^{ab}))$, fields on a 3-manifold Σ;

Vector Constraint:

$$H[N^a] := \int_\Sigma N^a \left\{-2q_{ab}\nabla_c \pi^{cb}\right\}$$

Scalar Constraint:

$$H[N] := \int_\Sigma N \left\{-\sqrt{q}\ R(q) + \frac{1}{\sqrt{q}}(\pi^{ab}\pi_{ab} - \frac{1}{2}(\pi^c_c)^2)\right\}$$

Connection Formulation of GR

Canonical fields:

SU(2) Connection $\quad A^i_a \quad (= \gamma_{BI}(K^i_a - \Gamma^i_a))$,

Densitized Triad $\quad E^a_i := \gamma_{BI} P^a_i \quad (= \sqrt{q} e^a_i,\ q_{ab} := e^i_a e^j_b \delta_{ij})$;

Gauss Constraint: $G[\Lambda] := \int_\Sigma \Lambda^i \left[\partial_a P^a_i + \epsilon^k_{ij} A^j_a P^a_k\right]$

Diffeo Constraint: $H[N^a] := \int_\Sigma N^a \left[F^i_{ab} P^b_i - A^i_a G_i\right]$

Hamiltonian Constraint:

$$H[N] := \tfrac{\kappa\gamma}{2} \int_\Sigma N \left[\frac{P^a_i P^b_j}{\sqrt{q}} \left\{\epsilon^{ij}{}_k F^k_{ab} - 2(1+\gamma^2) K^i_{[a} K^j_{b]}\right\}\right]$$

Choice of Basic Variables for quantization

Usual ${}^3A_f := \int d^3x A_a^i f_i^a$ is not suitable because:

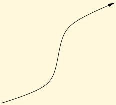
(i) The variables are non-covariant under gauge transformations;

(ii) polynomials in these variables are not integrable with respect to any of the known diffeomorphism invariant measures.

(These are non-issues for usual gauge theories in fixed space-times and for perturbative analyses.)

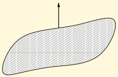
Holonomy-Flux Variables

Following lattice gauge theories, choose Holonomies as basic configuration space variables,



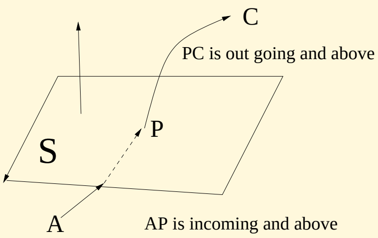
$$h_e(A) := P\exp \int_e A^i \tau_i$$

In order to act invariantly on the polynomials of holonomies, the momentum variables are required to be the Fluxes,



$$E_{S,f} := \int_S ds^{ab} \mathcal{E}_{abc} E^c_i f^i$$

Holonomy-Flux Poisson Brackets



$$\{E_{S,f}, h_e\} = \frac{1}{2}\sum_{e_p} k(e_p) f^i(p) \begin{cases} h_{e_p}\tau_i & \text{if } e_p \text{ is out-going} \\ -\tau_i h_{e_p} & \text{if } e_p \text{ is in-coming} \end{cases}$$

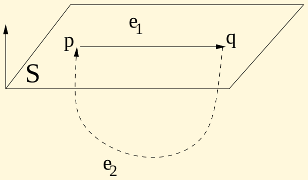
$$k(e_p) = \begin{cases} 0 & e_p \text{ is tangential to } S \text{ , or } e \cap S = \emptyset \\ 1 & e_p \text{ lies above } S \\ -1 & e_p \text{ lies below } S \end{cases}$$

Non-Commutativity of Fluxes

The flux-flux Poisson bracket however is not zero. This can be seen from, ($T_\alpha := \mathrm{Tr}(h_\alpha)$, $\alpha := e_2 \circ e_1$) ,

$$\{\mathrm{T}_\alpha, \{E_{S,f}, E_{S,g}\}\} = \frac{1}{4}\mathrm{Tr}(\tau_k h_\alpha)\epsilon^k{}_{ij}\left[f^i(p)g^j(p) - f^i(q)g^j(q)\right]$$

$$\neq 0$$



Loop Quantization: Kinematical Hilbert Space

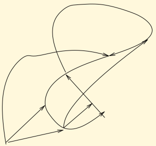
Construct a Hilbert space on which the holonomies are represented multiplicatively (eg as a representation of the C* algebra of holonomies).

Demand that the spatial diffeomorphisms are represented by unitary operators. Uniqueness Results of (L.O.S.T.) show:

There is only one such Hilbert space. It consists of wave functions of N, arbitrary but finite, number of holonomies. The holonomies are SU(2) group elements and the integration measure is the Haar measure on N-fold products of SU(2).

Kinematical Hilbert Space (Cont. . .)

The space is conveniently described in terms of an orthonormal basis provided by the Spin Network functions.



Closed, piece-wise analytic, oriented graph γ. With each edge associate a representation π_e of SU(2). With each vertex associate an intertwiner C_v. Then,

$$T_{\gamma,\vec{\pi}_e,\vec{C}_v}(A) \; := \; \text{``}Tr\text{''} \left(\prod_e \pi_e(h_e(A)) \prod_v C_v \right)$$

Kinematical Hilbert Space (Cont...)

$$\begin{aligned}\langle T_{\gamma,\vec{\pi}_e,\vec{C}_v}, T_{\gamma' \ , \ \vec{\pi'}_e,\vec{C'}_v}\rangle &:= \int_{\mathcal{A}} (d\mu_{\mathrm{Haar}})^{E(\gamma)} T^*_{\gamma,\vec{\pi}_e,\vec{C}_v} T_{\gamma',\vec{\pi'}_e,\vec{C'}_v} \\ &= \delta_{\gamma,\gamma'}\delta_{\vec{\pi},\vec{\pi'}}\delta_{\vec{C},\vec{C'}}\end{aligned}$$

These functions are SU(2) gauge invariant. Under a diffeomorphism, graphs are moved but not the labels π_e, C_v.

General wave functions are linear combinations of spin network functions. The action of the flux operators is given by,

$$\hat{P}_{S,f}\Psi(A) = \frac{\hbar}{2}\sum_{v\in S} f^i(v) \sum_{e \ at \ v} k(S,e)\hat{J}_i^{\ v,e} \ \Psi(A)$$

$\hat{J}$'s are left(right) invariant vector fields on SU(2).

Quantum Geometry

In quantum theory, there is an operator associated with geometrical quantities eg area of a surface, volume of region, length of curve etc. The spectra of these operators are discrete and in this specific sense, quantum geometry is discrete. The manifold is not discretised.

Spectrum of Area operator:

$$\begin{aligned}
\hat{A}_S|T_{\gamma,\vec{\pi},\vec{c}}\rangle &= 4\pi\gamma\ell_{\mathrm{P}}^2 \sum_{v\in\gamma\cap S} \lambda_v |T_{\gamma,\vec{\pi},\vec{c}}\rangle \\
\lambda_v^2 &= 2j_u(j_u+1) + 2j_d(j_d+1) - j_{u+d}(j_{u+d}+1), \\
& \quad j's \in \mathbb{Z}/2
\end{aligned}$$

Black Hole Entropy

The idea is that black hole entropy arises from excitations of quantum geometry of horizon with area A. We know that area of a surface arises from spin-networks impinging on the surface.

How is a surface describing a 'black hole horizon' distinguished? Does the counting of states give entropy proportional to its area, at least for macroscopic black holes?

Classically, black hole horizons are distinguished by certain boundary conditions satisfied by the fields on the horizon. All known stationary black hole horizons satisfy these and there is an infinite class of non-stationary space-times containing isolated horizons satisfying these conditions.

Black Hole Entropy (Cont . . .)

Isolated horizon is a null hyper-surface $\Delta \sim \mathbb{R} \times S^2$ whose intrinsic geometry is determined by $A^i r_i$ on S^2 and the following condition holds on S^2.

$$F \ := dW = -\frac{2\pi}{a_0}\kappa\Sigma^i r_i \quad , \quad W := \underline{A}^i r_i \quad , \quad \Sigma^i_{ab} := \mathcal{E}_{abc}\underline{E}^{c\,i}$$

In the quantum theory, this equation is promoted to an operator equation. Solutions of this equation which give area a_0 within a margin of Δa, are identified as black hole states.

Counting states after tracing over bulk states, the entropy, for large a_0, matches with the Bekenstein-Hawking entropy for a particular value of γ. This holds for all types of horizons including the extremal ones.

Big Bang Singularity: Classical View

$$ds^2 \;:=\; -dt^2 + a^2(t)\left\{dr^2 + r^2 d\Omega^2\right\}$$

$$\begin{aligned} H &= \left[-\frac{2\pi G}{3}\frac{p_a^2}{V_0 a}\right] + \left[\frac{1}{2}\frac{p_\phi^2}{a^3 V_0}\right], \; c := V_0^{1/3}\dot{a}\gamma \;, \; |p| := V_0^{2/3}a^2 \\ &= \left[-\frac{3}{\kappa}\left(\gamma^{-2}c^2\sqrt{|p|}\right)\right] + \left[\frac{1}{2}|p|^{-3/2}p_\phi^2\right] \;, \; \kappa := 8\pi G \;. \end{aligned}$$

$$c \;=\; \pm\gamma\sqrt{\frac{\kappa}{6}}\frac{|p_\phi|}{|p|} \quad , \quad \dot{p} \;=\; \pm 2\sqrt{\frac{\kappa}{6}}|p_\phi||p|^{-1/2}$$

$$\dot{\phi} \;=\; p_\phi|p|^{-3/2} \quad , \quad \dot{p_\phi} \;=\; 0 \;,$$

$$\frac{dp}{d\phi} \;=\; \pm\sqrt{\frac{2\kappa}{3}}|p| \quad \Rightarrow \quad \mathbf{p}(\phi) \;=\; \mathbf{p}_*\mathbf{e}^{\pm\sqrt{\frac{2\kappa}{3}}(\phi-\phi_*)}$$

Loop Quantum Cosmology

Holonomy-Flux representation:

$$\hat{p}|\mu\rangle = \frac{1}{6}\gamma\ell_{\mathrm{P}}^2\mu|\mu\rangle \ , \ \langle\mu|\mu'\rangle = \delta_{\mu,\mu'} \ , \ \mu \in \mathbb{R}$$

$$\widehat{h_\nu}|\mu\rangle := \widehat{e^{\frac{i}{2}\nu c}}|\mu\rangle = |\mu+\nu\rangle$$

This implies that inverse triad operator must be defined in a different manner and the 'connection' c must be replaced by $\sin(\mu_0 c/2)$.

These render both the terms in the Hamiltonian bounded, thereby implying a 'bounce'.

Singularity Resolution

More precisely, Dirac observables as well as the physical semi-classical states are explicitly constructed. By taking expectation values of suitable Dirac operator (analogue of $p(\phi)$) in suitable semi-classical states (large universe), the analogue of the $p(\phi)$ curve is constructed which shows that:

The expectation value follows the classical curve closely till the volume becomes a few times the Planck volume; The uncertainties grow, invalidating the classical picture and the curve deviates from the contracting branch; It emerges and tracks the expanding $p(\phi)$ curve (Ashtekar-Pawlowski-Singh).

The Big Bang Singularity is resolved!

LQC Effective picture and implications

An effective Hamiltonian which incorporates the holonomy and the inverse triad corrections can be constructed which reproduces the exact picture pretty well and can be used directly for phenomenology.

$$
\begin{aligned}
H &= -\frac{3}{\kappa}\sqrt{p}\left[\frac{\sin^2\left(\epsilon(p)c\right)}{\gamma^2\epsilon^2(p)}\right] + \frac{1}{2}F_{j,l}(p)p_\phi^2 \\
\epsilon(p) &:= \Delta\sqrt{\gamma}\ell_{\mathrm{P}}p^{-1/2} \\
F_{j,l}(p) &\rightarrow \begin{cases} p^{-3/2} & p \rightarrow \infty \\ p^{3/(1-l)} & p \rightarrow 0 \end{cases}
\end{aligned}
$$

Genericness of inflation as well as adequate e-foldings are deduced from such effective pictures after adding potential terms.

Summary and Open Issues

The twin minimal expectations are met. Homogeneous, isotropic singularities are resolved and indications are that some of the non-isotropic ones are resolved as well. While black hole entropy is well understood for the entire class of isolated horizons, Hawking effect is yet to be computed.

Among the formal developments, Hamiltonian Constraint is still to be understood;

Semi-classical sector is not under control yet;

Preliminary work on Matter Couplings is available but needs to be looked at in more detail.

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Thank You