Fluid Dynamics from Gravity

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- It has long been argued that $U(N)$ gauge theories reduce to effectively classical systems in the t' Hooft large N limit.
- $\mathcal{N} = 4$ Yang Mills theories are $U(N)$ gauge theoeries. These theories are conformally invariant; they define a line of fixed points labeled by a continuous coupling constant λ. In 1997 Maldacena identified the corresponding large N classical systems.
- While the classical equations identified by Maldacena are unfamliar (and appear complicated) at finite λ, they simplify dramatically at large λ. In this limit they reduce to the equations of Einstein (IIB super) gravity on spacetimes that asymptote to $AdS_5 \times S^5$.

- The gravitational description of field theory dynamics is unfamiliar partly because it applies only at very strong coupling.
- Even at strong coupling, however, we have some qualitative expectations of local QFTs. For instance they are expected to equilibriate at every finite temperature.
- What is the gravitational description of this thermal state? Answer (Witten): an asymptotically AdS black brane.
- This answer is universal in the following sense. Every 2 derivative theory of gravity interacting with other fields of spin ≤ 2 admits a consistent truncation to Einstein's equations with a negative cosmological constant. Black brane solutions lie in this universal sector.

- In appropriately chosen units, Einstein's equations with a negative cosmological constant in $d+1$ dimensions are

$$R_{MN} - \frac{R}{2} g_{MN} = \frac{d(d-1)}{2} g_{MN} : \quad : M, N = 1 \ldots d+1$$

- The black brane at temperature T and velocity u_μ are a d parameter set of exact solutions of these equations

$$ds^2 = \frac{dr^2}{r^2 f(r)} + r^2 \mathcal{P}_{\mu\nu} dx^\mu dx^\nu - r^2 f(r) u_\mu u_\nu dx^\mu dx^\nu$$

$$f(r) = 1 - \left(\frac{4\pi T}{d\ r}\right)^d ; \quad \mathcal{P}_{\mu\nu} = g_{\mu\nu} + u_\mu u_\nu$$

- These solutions have a horizon at $r = \frac{4\pi T}{d}$. The thermal nature of these solutions follows from well known properties of event horizons.

- Einstein's equations allow us to study deviations from thermal equilibrium. Natural first question: what is the spectrum of linearized fluctuations about thermal equilibrium?.
- If we impose the requirement of regularity of the future horizon, the anwer is given by gravitational 'quasinormal modes'. Discrete infinity of such modes labeled by integers. For the n^{th} mode $\omega = \omega_n(k)$. Frequency complex corresponding to decay.
- It follows from conformal invariance that $\omega_n(0) = \frac{f(n)}{T}$. $f(n) \neq 0$ except for the 4 Goldstone modes corresponding to variations of T and u_μ. Infact Policastro Starinets and Son demonstrated that the dispersion relation for these Goldstone modes at small k takes the form predicted by fluid dynamics -(shear and sound waves) provided $\frac{\eta}{s} = \frac{1}{4\pi}$

- So we now have a first hint that black branes mimic the behaviour of thermal QFTs for dynamical, not just static purposes.
- Can we take this further? It is well known that the effective dynamical description of field theories in *local* thermal equilibrium are the equations of fluid dynamics.
- Fluid dynamics is a derivative expansion: it works provided all variations are slow (compared to a dynamical relaxation time) but does not require amplitude variations to be small.
- Thus the AdS/CFT correspondence appears to implythat the equations of asymptotically *AdS* gravity reduce to (relativistic generalizations of) the Navier Stokes equations at the *full nonlinear level* in an appropriate long distance expansion. Is this exciting suggestion true?

- It is useful to isolate the trivial from the nontrivial parts of this suggestion. The equations of uncharged fluid dynamics take the form $\partial^\mu T_{\mu\nu} = 0$. These equation follows simply from symmetries that are true on both sides of the duality, and are trivially automatic on both sides.
- However $\partial^\mu T_{\mu\nu} = 0$ are d equations for $\frac{d(d+1)}{2} - 1$ variables. They constitute a well defined dynamical system only if you are able to express $T^{\mu\nu}$ as a function of d variables. This 'constitutive relationship' is the nontrivial assertion of fluid dynamics.
- How could this work in gravity? Suggestion: lets use the collective coordinate method (or Goldstone philosophy) on the d parameter set of exact black brane solutions.

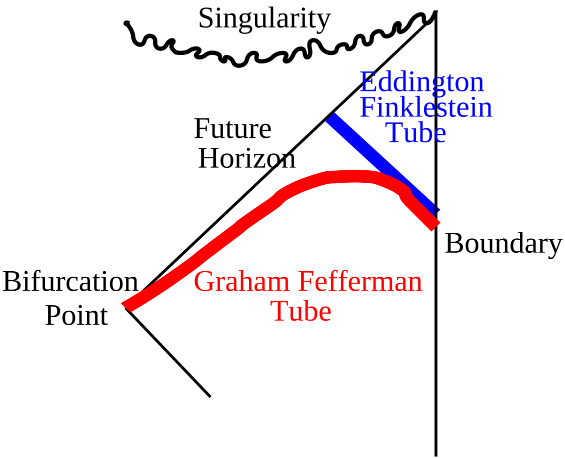
- We look for solutions that 'locally' approximate black branes but with space varying velocities and temperatures. More precisely we search for bulk solution tubewise approximated by black branes. But along which tubes?
- Naive guess: lines of constant x^μ in Schwarschild (Graham Fefferman) coordinates, i.e. metric approximately

$$ds^2 = \frac{dr^2}{r^2 f(r)} + r^2 \mathcal{P}_{\mu\nu}(x) dx^\mu(x) dx^\nu(x) - r^2 f(r) u_\mu u_\nu dx^\mu dx^\nu$$

$$f(r) = 1 - \left(\frac{4\pi T(x)}{d\ r}\right)^d ; \quad \mathcal{P}_{\mu\nu} = g_{\mu\nu}(x) + u_\mu(x) u_\nu(x)$$

- Does not seem useful. Appears to be a bad starting point for perturbation theory. Also has several interpretative difficulties.

Penrose diagram



- Causality suggests the use of tubes centered around ingoing null geodesics. In particular we try

$$
\begin{aligned}
ds^2 = g^{(0)}_{MN} dx^M dx^N = & -2u_\mu(x) dx^\mu dr + r^2 \mathcal{P}_{\mu\nu}(x) dx^\mu dx^\nu \\
& - r^2 f(r, T(x)) u_\mu(x) u_\nu(x) dx^\mu dx^\nu
\end{aligned}
$$

- Metric generally regular but not solution to Einstein's equations. However solves equations for constant $u^\mu, T, g_{\mu\nu}$. Consequntly appropriate starting point for a perturbative soln of equations in the parameter $\epsilon(x)$.

- That is we set

$$g_{MN} = g^{(0)}_{MN}(\epsilon x) + \epsilon g^{(1)}_{MN}(\epsilon x) + \epsilon^2 g^{(2)}_{MN}(\epsilon x) \ldots$$

  and attempt to solve for $g^{(n)}_{MN}$ order by order in ϵ.
- Perturbation expansion surprisingly simple to implement. Nonlinear partial differential equation $\to \frac{d(d+1)}{2}$ ordinary differential equations, in the variable r at each order and each boundary point.

- It turns out that all equations can be solved analytically (and rather simply). Upon solving the equations we find that the perturbative procedure spelt out above can be implimented at n^{th} order *only* when an integrability condition of the form $\partial_\mu T^{\mu\nu}_{n-1}(u^\mu(x), T(x))$ where $T^{\mu\nu}_{n-1}(u^\mu(x), T(x))$ is a specific function of temperature and velocities and their spacetime derivatives to $(n-1)^{th}$ order. This function is determined directly from Einstein's equations by the perturbative procedure.
- For every $u^\mu(x)$ and $T(x)$ that satisfies this Fluid Dynamical equation we have a solution to Einstein's equations. The map from fluid dynamics to gravity is locally invertable assuming regularity of the future event horizon.

Explicit Results at second order

We have explicitly implemented our perturbation theory to second order.

$$
\begin{aligned}
ds^2 &= -2u_\mu dx^\mu \left(dr + r\, A_\nu dx^\nu\right) + r^2 g_{\mu\nu} dx^\mu dx^\nu \\
&- \left[\omega_\mu{}^\lambda \omega_{\lambda\nu} + \frac{1}{d-2}\mathcal{D}_\lambda \omega^\lambda{}_{(\mu} u_{\nu)} - \frac{1}{d-2}\mathcal{D}_\lambda \sigma^\lambda{}_{(\mu} u_{\nu)} \right. \\
&\left. + \frac{\mathcal{R}}{(d-1)(d-2)} u_\mu u_\nu \right] dx^\mu dx^\nu \\
&+ \frac{1}{(br)^d}\left(r^2 - \frac{1}{2}\omega_{\alpha\beta}\omega^{\alpha\beta}\right) u_\mu u_\nu dx^\mu dx^\nu \\
&+ 2(br)^2 F(br) \left[\frac{1}{b}\sigma_{\mu\nu} + F(br)\sigma_\mu{}^\lambda \sigma_{\lambda\nu}\right] dx^\mu dx^\nu \ldots
\end{aligned}
$$

Explicit Results at second order

$$
\begin{aligned}
&- 2(br)^2\frac{\sigma_{\alpha\beta}\sigma^{\alpha\beta}}{d-1}P_{\mu\nu}K_1(br) - \frac{u_\mu u_\nu}{(br)^{d-2}}\frac{\sigma_{\alpha\beta}\sigma^{\alpha\beta}}{(d-1)}K_2(br)\\
&+ \frac{2\,L(br)}{(br)^{d-2}}\left[P_\mu^\lambda \mathcal{D}_\alpha \sigma^\alpha{}_\lambda u_\nu + P_\nu^\lambda \mathcal{D}_\alpha \sigma_\lambda{}^\alpha u_\mu\right] dx^\mu dx^\nu\\
&- 2(br)^2 H_1(br)\left[u^\lambda \mathcal{D}_\lambda \sigma_{\mu\nu} + \sigma_\mu{}^\lambda \sigma_{\lambda\nu} - \frac{\sigma_{\alpha\beta}\sigma^{\alpha\beta}}{d-1}P_{\mu\nu}\right.\\
&\qquad \left. + C_{\mu\alpha\nu\beta}u^\alpha u^\beta\right] dx^\mu dx^\nu\\
&+ 2(br)^2 H_2(br)\left[u^\lambda \mathcal{D}_\lambda \sigma_{\mu\nu} + \omega_\mu{}^\lambda \sigma_{\lambda\nu} - \sigma_\mu{}^\lambda \omega_{\lambda\nu}\right] dx^\mu dx^\nu
\end{aligned}
$$

Explicit results at second order

Where

$$F(br) \equiv \int_{br}^{\infty} \frac{y^{d-1}-1}{y(y^d-1)} dy \ ; \ L(br) \equiv \int_{br}^{\infty} \xi^{d-1} d\xi \int_{\xi}^{\infty} dy \ \frac{y-1}{y^3(y^d-1)}$$

$$H_2(br) \equiv \int_{br}^{\infty} \frac{d\xi}{\xi(\xi^d-1)} \int_{1}^{\xi} y^{d-3} dy \left[1+(d-1)yF(y)+2y^2F'(y)\right]$$

$$K_1(br) \equiv \int_{br}^{\infty} \frac{d\xi}{\xi^2} \int_{\xi}^{\infty} dy \ y^2 F'(y)^2 \ ; \ H_1(br) \equiv \int_{br}^{\infty} \frac{y^{d-2}-1}{y(y^d-1)} dy$$

$$K_2(br) \equiv \int_{br}^{\infty} \frac{d\xi}{\xi^2} \left[1-\xi(\xi-1)F'(\xi)-2(d-1)\xi^{d-1} + \left(2(d-1)\xi^d-(d-2)\right)\int_{\xi}^{\infty} dy \ y^2 F'(y)^2\right]$$

The dual stress tensor corresponding to this metric is given by $(4\pi T = b^{-1}d)$

$$
\begin{aligned}
T_{\mu\nu} &= p\left(g_{\mu\nu} + d u_\mu u_\nu\right) \\
&\quad - 2\eta\left[\sigma_{\mu\nu} - \tau_\pi u^\lambda \mathcal{D}_\lambda \sigma_{\mu\nu} - \tau_\omega\left(\sigma_\mu{}^\lambda \omega_{\lambda\nu} - \omega_\mu{}^\lambda \sigma_{\lambda\nu}\right)\right] \\
&\quad + \xi_\sigma\left[\sigma_\mu{}^\lambda \sigma_{\lambda\nu} - \frac{\sigma_{\alpha\beta}\sigma^{\alpha\beta}}{d-1} P_{\mu\nu}\right] + \xi_C C_{\mu\alpha\nu\beta} u^\alpha u^\beta
\end{aligned}
$$

$$
p = \frac{1}{16\pi G_{d+1} b^d} \quad ; \quad \eta = \frac{s}{4\pi} = \frac{1}{16\pi G_{d+1} b^{d-1}}
$$

$$
\tau_\pi = (1 - H_1(1))b \quad ; \quad \tau_\omega = H_1(1)b \quad ; \quad \xi_\sigma = \xi_C = 2\eta b
$$

- Note that gravity reduces to fluid dynamics with particular (holographically determined) values for dissipative parameters. As we have seen the schematic form of the fluid stress tensor is

$$T_{\mu\nu} = aT^d(g_{\mu\nu} + du_\mu u_\nu) + bT^{d-1}\sigma_{\mu\nu} + T^{d-2}\sum_{i=1}^{5} c_i S^i_{\mu\nu}$$

- a is a thermodynamic parameter. b is related to the viscosity: we find $\eta/s = 1/(4\pi)$. c_i coefficients of the five traceless symmetric Weyl covariant two derivative tensors are second order transport coefficients. Value disagree with the predictions of the Israel Stewart formalism.
- Recall that results universal. Should yield correct order of magnitude estimate of transport coefficients in any strongly coupled CFT.

- Our solutions are singular at $r = 0$. Quite remarkably it is possible (under certain conditions) to demonstrate that these solutions have event horizons and to explicitly determine the event horizon manifold order by order in the derivative expansion. This horizon shields the $r = 0$ singularity from the boundary.
- Our control over the event horizon, together with the classic area increase theorem of general relativity, can be used to derive an 'entropy current' for our fluid flows that is local and has positive divergence.

Entropy Current at second order

Explicitly this entropy current is given to second order by

$$4\, G_{d+1}\, b^{d-1}\, J_S^{\mu} = [\, 1 + b^2 \,(A_1\, \sigma^{\alpha\beta}\, \sigma_{\alpha\beta} + A_2\, \omega^{\alpha\beta}\, \omega_{\alpha\beta} + A_3\, \mathcal{R}\,)\,]\, u^{\mu} + b^2\, [\, B_1\, \mathcal{D}_{\lambda}\, \sigma^{\mu\lambda} + B_2\, \mathcal{D}_{\lambda}\, \omega^{\mu\lambda}\,]$$

where

$$A_1 = \frac{2}{d^2}(d+2) - \frac{K_1(1)d + K_2(1)}{d}\,, \qquad A_2 = -\frac{1}{2d}, \qquad B_2 = \frac{1}{d-2}$$

$$B_1 = -2A_3 = \frac{2}{d(d-2)}$$

- As we have seen above, gravity completely and unambiguously determines every term in the expansion of a universal fluid dynamical stress tensor.
- However once we move away from the universal sector - e.g. by turning on Maxwell fields and scalars - the results for arbitrary components of the fluid stress tensor change; however it turns out that some ratios of the coefficients of different terms in this stress tensor remain unchanged under these deformations (Haack and Yarom). Remenicent of $\frac{\eta}{s}$. Should be investigated further.

- Several people have generalized the basic story described above in several directions. First the story can be generalized to nonconformal situations. As a concrete example the gravitational dual of the hydrodynamical description of the theory on the world volume of p branes at finite temperature (for all p)
- This turns out to be easier than expected for a remarkable reason. It turns out that every Dp backgroud admits a consistent truncation to the metric/dilaton sector. Further this metric dilaton system may formally be regarded as the dimensional reduction of Einstein gravity with a negative cosmological constant in a fictitious higher dimension.
- For the case of the fundamental string and the D4 branes this follows by descent from M theory. Howeve the observation is always true. This allows the immediate determination of the relevant hydrodynamical behaviours using the AdS results described above.

- Another simple generalization is the study of fluids on an arbitrary (and not flat) boundary space. This permits the study of effectively forced fluids.
- Perhaps the most interesting generalization, however, is to the study of charged fluids. Charged fluids have a new conservation law; in order to study these systems we need to add Maxwell fields to the bulk. The Einstein Maxwell system has a well known set of charged black brane solutions. It is possible to replay the story described above on the Einstein- Maxwell system, allowing the charge density (as well as the velocity and temperature) of the brane to vary as a function of x.

- The end result is a map between the solutions of charged fluid dynamics and those of the long distance Einstein Maxwell system. In $d = 4$ has also been done in the presence of a background magnetic field.
- The procedure yields expressions for the stress tensor and charge currents as a function of local temperatures, velocities and chemical potentials.
- We find a surprise here even at first order in the derivative expansion. In addition to the usual diffusive currents, in $d = 4$ we find a term in the charge current proportional to $\epsilon_{\mu\nu\rho\sigma}\omega^{\nu\rho}u^{\sigma}$. This is important because this term was ignored by Landau and Lifshitz and perhaps all authors subsequently.

- It turns out that this new term (which violates parity but not CP) arises only if the bulk dual has a Chern Simons term in addition to the Maxwell term for gauge fields.
- Landau Lifshitz ignored this term, perhaps because they thought that it would be impossible to construct a point wise increasing entropy current for fluid equations with such a term.
- However the gravitational construction is automatically equiped with a satisfactory local entropy current, so that cannot be correct. The story here has been clarrfied recently by Son and Surowka. It turns out that such a term is required by a $U(1)^3$ anomaly; the coefficient of this term is determined by the anomaly coefficient, independent of holography, and so may have experimental consequences.

- It has recently been realized that there exist asymptotically *AdS* charged black brane solutions immersed in the condensate of a charged scalar field.
- It is possible to replay the story described above (some work done, more under progress) to obtain the gravitational dual description of superfluid dynamics. A possible window of opportunity.

- We have so far studied the gravity dual of locally equilibrated fluids. The gravity dual of the process of the formation of equlibriation is the process of black hole (or black brane) formation.
- It is interesting that this process of black hole formation can be studied analytically in some limits. This allows a uniform gravity description of the forcing of a field theory (initially in its vacuum), its local equilibriation, and then the subsequent fluid dynamical motion of this equilibriated fluid.
- The gravitational equilibriation process can sometimes be tuned to exhibit a Choptwick type singularity. All of this suggests that gravity can contribute significantly to the study of the attainment of equilibrium in field theories. It would be great to have a sharp uinversal question to which to apply these techniques.

- Asymptotically AdS_{d+1} gravity reduces, in the long wavelength limit, to the equations $d+1$ dimensional Navier Stokes equations with gravitationally determined dissipative parameters.
- The gravitationaly determined fluid dynamical system has no free parameters, and so enjoys a degree of universality. Values of these parameters are interesting and sometimes surprising.
- This map allows us to reword open problems in fluid dynamicsas problems in gravity. Potential for new insights?
- It would be very interesting to extend this connection past the large N (or classical) limit.