Phenomenological Analysis of Neutrino Mass Matrices with a Texture Zero and a Vanishing Minor and their CP-Odd Weak Basis Invariants

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- During past several years enormous progress has been made in the determination of neutrino masses and mixings and in the study of neutrino mass matrix

- One of the main theoretical challenge is to understand the dynamics behind the observed pattern of the neutrino masses and mixings

Neutrino Mass Matrix

The main objectives of the neutrino physics include

- The determination of the absolute mass scale of the neutrinos
- Their mass spectra/ mass hierarchy
- Subdominant structure of mixing, namely, 1-3 mixing, deviation of 2-3 mixing from maximality and the CP violating phases.

Neutrino Mass Matrix

- The mass matrix for Majorana neutrinos contains nine physical parameters
- Viz. the three mass eigenvalues, three mixing angles and the three CP-violating phases .
- The two mass-squared differences and two mixing angles have been measured in Solar, Atmospheric and Reactor experiments.

Neutrino Mass Matrix

- The third mixing angle Θ_{13} and Dirac type CP violating phase δ are expected to be measured in the forthcoming neutrino oscillation experiments.

- Neutrino mass scale will be independently determined by the direct beta decay searches and cosmological observations.

Neutrino Mass Matrix

- Thus the neutrino mass matrix is not well determined
- Several attempts have been made in literature to restrict the form of neutrino mass matrices
  - Texture Zeros
  - Hybrid Textures
  - Traceless
  - Zero Determinant
  - Vanishing Minors

Seesaw Mechanism

- The seesaw mechanism is regarded as the prime candidate for understanding the scale of neutrino masses not only due to its simplicity but also due to its theoretical appeal

$$M_\nu = -M_D M_R^{-1} M_D^T,$$

where M_D is the Dirac and M_R is the right handed Majorana mass matrix.

Vanishing Minors

- Texture zeros in M_ν have been extensively studied.

  S. Dev, Sanjeev Kumar, Surender Verma and Shivani Gupta, Phys. Rev. D 76, 013002 (2007) [arXiv:hep-ph/0612102].

- It has been noted by many authors that the zeros of M_D and M_R are the progenitors of zeros in M_ν.

  A. Kageyama, S. Kaneko, N. Shimoyama and M. Tanimoto, Phys. Lett. B 538,96 (2002); L. Lavoura, Phys. Lett. B 609, 317 (2005); E. Ma Phys. Rev. D 71, 111301 (2005)

- However the zeros of M_D and M_R may not only show as zeros in the effective neutrino mass matrix M_ν.

Vanishing Minors

- Another interesting possibility is that these zeros show as vanishing minors in M_ν.

- The case where the zeros of M_R show as a vanishing minor in M_ν for diagonal M_D has been examined recently.

E. I. Lashin and N Chamoun, Phys. Rev. D 78, 073002 (2008); Phys. Rev. D 80, 093004 (2009)

Texture Zero and a Vanishing Minor

- We explore the more general possibility of simultaneous existence of a texture zero and a vanishing minor in M_ν. Both M_R and M_D are taken non diagonal and their zeros propagate via seesaw as a zero and a vanishing minor in M_ν.
- There are 36 texture structures of this type, 21 of which reduce to two texture zero cases.
- Of the remaining 15 textures only six are allowed by the current neutrino oscillation data.

Thirty-six allowed texture structures of M_ν with a texture zero and a vanishing minor

		A	B	C	D	E	F
1	$\begin{pmatrix} 0 & b & c \\ b & d & e \\ c & e & f \end{pmatrix}$	$df - e^2 = 0$	$bf - ec = 0$	$be - cd = 0$	Two zero	Two zero	Two zero
2	$\begin{pmatrix} a & 0 & c \\ 0 & d & e \\ c & e & f \end{pmatrix}$	$df - e^2 = 0$	Two zero	Two zero	$af - c^2 = 0$	Two zero	Two zero
3	$\begin{pmatrix} a & b & 0 \\ b & d & e \\ 0 & e & f \end{pmatrix}$	$df - e^2 = 0$	Two zero	Two zero	Two zero	Two zero	$ad - b^2 = 0$
4	$\begin{pmatrix} a & b & c \\ b & 0 & e \\ c & e & f \end{pmatrix}$	Two zero	$bf - ec = 0$	Two zero	$af - c^2 = 0$	$ae - bc = 0$	Two zero
5	$\begin{pmatrix} a & b & c \\ b & d & 0 \\ c & 0 & f \end{pmatrix}$	Two zero	Two zero	Two zero	$af - c^2 = 0$	Two zero	$ad - b^2 = 0$
6	$\begin{pmatrix} a & b & c \\ b & d & e \\ c & e & 0 \end{pmatrix}$	Two zero	Two zero	$be - dc = 0$	Two zero	$ae - bc = 0$	$ad - b^2 = 0$

S. Dev, Surender Verma ,Shivani Gupta and R. R. Gautam Phys. Rev. D 81, 053010 (2010) [arXiv:hep-ph/1003.1006].

Texture Zero and a Vanishing Minor

The simultaneous existence of a texture zero and a vanishing minor in the neutrino mass matrix gives following two conditions

$$M_{\nu(xy)} = 0,$$

$$M_{\nu(pq)}M_{\nu(rs)} - M_{\nu(tu)}M_{\nu(vw)} = 0.$$

$$\frac{m_1}{m_2}e^{-2i\alpha} = \frac{(-XA_1 - YA_2 + ZA_3 \pm \sqrt{X^2A_1^2 + (YA_2 - ZA_3)^2 - 2XA_1(YA_2 + ZA_3)})}{2XA_2}.$$

- $$\frac{m_1}{m_3}e^{-2i\beta} = -\frac{(XA_1 - YA_2 + ZA_3 \pm \sqrt{X^2A_1^2 + (YA_2 - ZA_3)^2 - 2XA_1(YA_2 + ZA_3)})}{2XA_3}e^{2i\delta}$$

$$\rho = \left|\frac{m_1}{m_3}e^{-2i\beta}\right|$$

$$\sigma = \left|\frac{m_1}{m_2}e^{-2i\alpha}\right|$$

$$\alpha = -\frac{1}{2}\arg\left(\frac{(-XA_1 - YA_2 + ZA_3 \pm \sqrt{X^2A_1^2 + (YA_2 - ZA_3)^2 - 2XA_1(YA_2 + ZA_3)})}{2XA_2}\right),$$

$$\beta = -\frac{1}{2}\arg\left(-\frac{(XA_1 - YA_2 + ZA_3 \pm \sqrt{X^2A_1^2 + (YA_2 - ZA_3)^2 - 2XA_1(YA_2 + ZA_3)})}{2XA_3}e^{2i\delta}\right)$$

where $X = U_{x1}U_{y1}$, $Y = U_{x2}U_{y2}$, $Z = U_{x3}U_{y3}$

$$A_h = (U_{pl}U_{ql}U_{rk}U_{sk} - U_{tl}U_{ul}U_{vk}U_{wk}) + (l \leftrightarrow k)$$

Results and Discussions

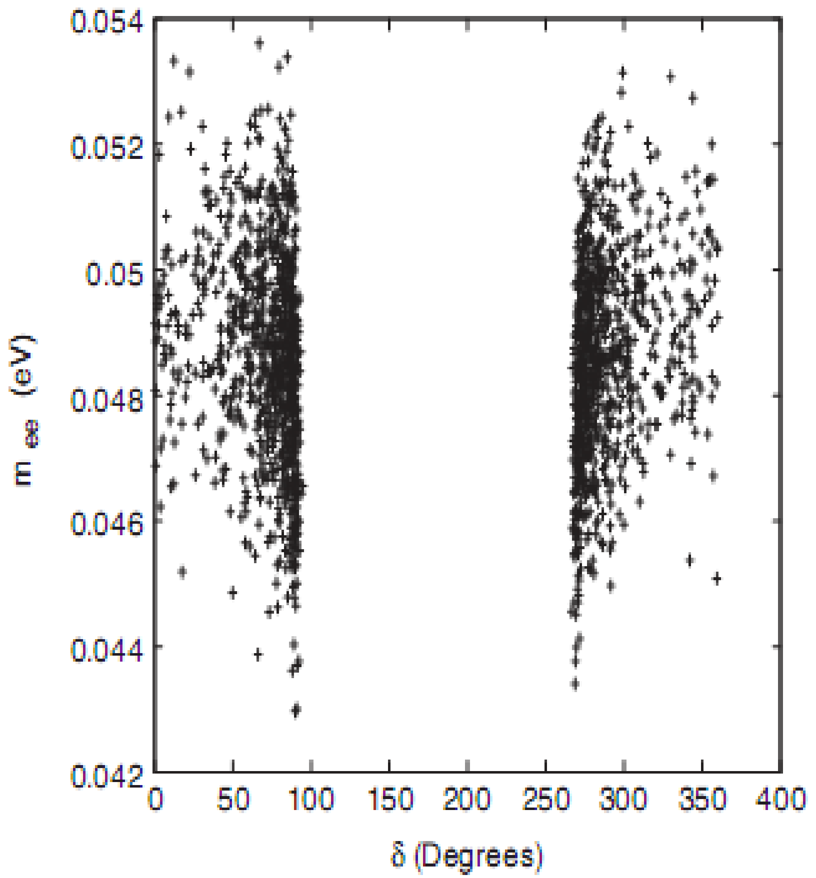
- Class 2A
- Texture structure has zero at (1,1) entry and vanishing minor corresponding to (1,1) entry.

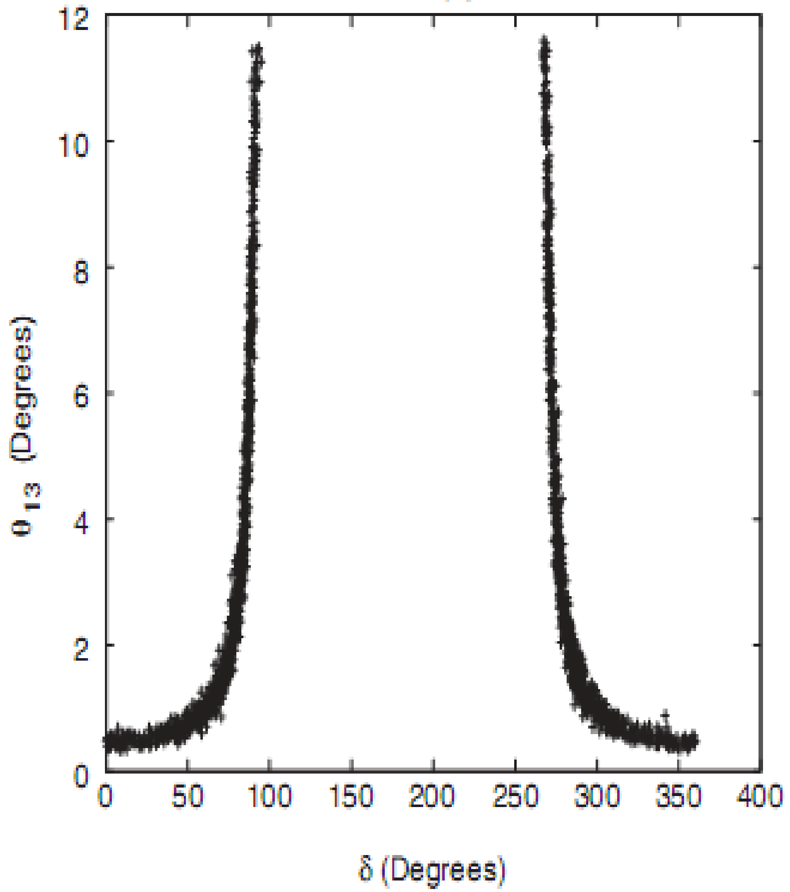
$$\rho = \left| \frac{m_1}{m_3} \right| \approx \frac{1}{s_{13}^2} + O\left(\frac{1}{s_{13}}\right),$$

$$\sigma = \left| \frac{m_1}{m_2} \right| \approx 1 - \frac{\cos\delta s_{13} s_{23}}{c_{12} c_{23} s_{12}} + O(s_{13}^2).$$

Inverted hierarchy and Cosδ is positive

Correlation plots for class 2A





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Phys. Rev. D 81, 053010 (2010) [arXiv:hep-ph/1003.1006].

Results and Discussions

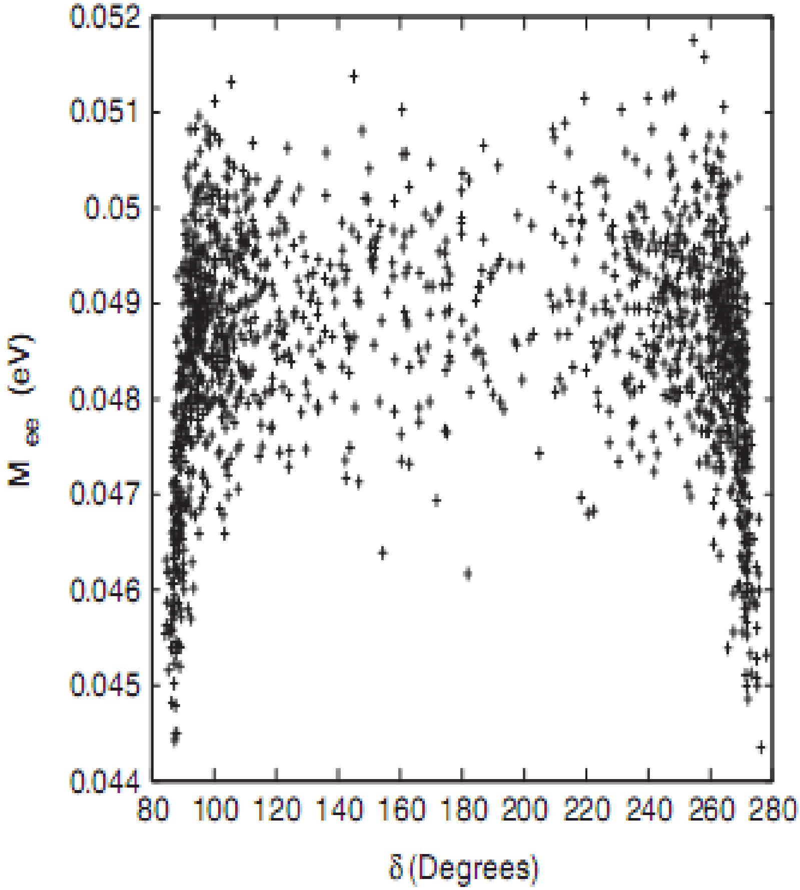
- Class 3A
- Texture structure has zero at (1,3) entry and vanishing minor corresponding to (1,1) entry.

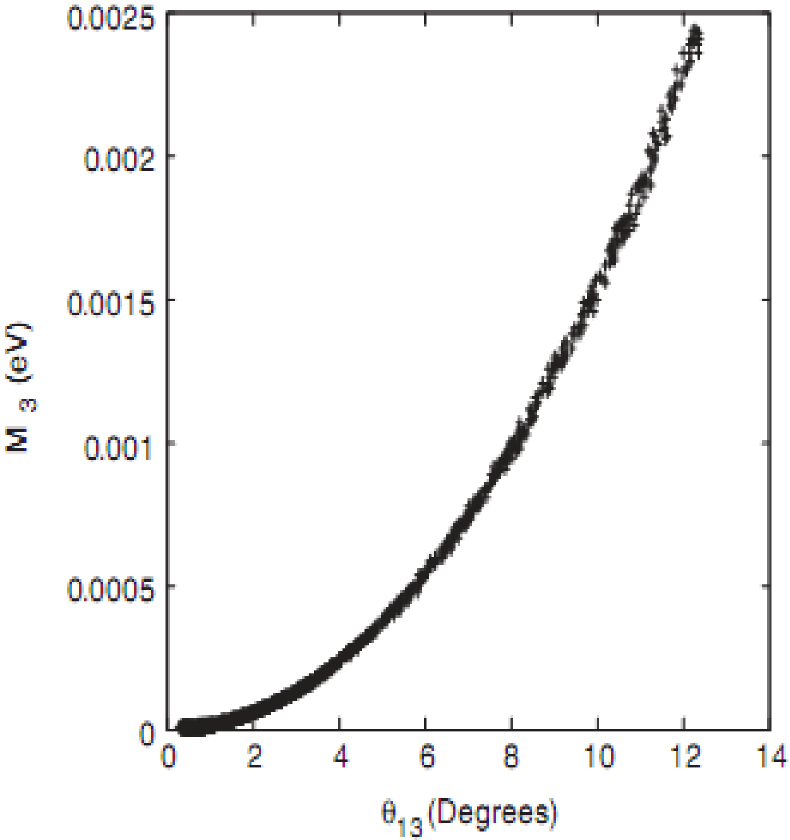
$$\rho = \left|\frac{m_1}{m_3}\right| \approx \frac{1}{s_{13}^2} + O\Big(\frac{1}{s_{13}}\Big),$$

$$\sigma = \left|\frac{m_1}{m_2}\right| \approx 1 + \frac{\cos\delta\, s_{13} c_{23}}{c_{12} s_{23} s_{12}} + O(s_{13}^2).$$

Inverted hierarchy and Cosδ is negative

Correlation plots for class 3A





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Phys. Rev. D 81, 053010 (2010) [arXiv:hep-ph/1003.1006].

Symmetry Realization

- All the phenomenologically viable structures of M_ν with a texture zero and a vanishing minor (2A, 3A, 2D, 3F, 4B and 6C) can be realized in a simple way in models based on Type-I seesaw mechanism with the discrete Abelian flavor symmetry ($Z_{12}XZ_2$).

- ***W. Grimus, A. S. Joshipura, L. Lavoura and M. Tanimoto, Eur. Phys. J C 36, 227 (2004);***
- ***S. Dev, Surender Verma ,Shivani Gupta and R. R. Gautam Phys. Rev. D 81, 053010 (2010) [arXiv:hep-ph/1003.1006].***

Symmetry Realization

$$M_D = \begin{pmatrix} 0 & 0 & a \\ b & c & 0 \\ 0 & d & e \end{pmatrix}, \qquad M_R = \begin{pmatrix} 0 & A & 0 \\ A & B & 0 \\ 0 & 0 & C \end{pmatrix}.$$

The resulting M_ν generated via the seesaw mechanism (Class 2D) takes the form

$$M_\nu = \begin{pmatrix} \frac{a^2}{C} & 0 & \frac{ae}{C} \\ 0 & \frac{b(-(bB)+2Ac)}{A^2} & \frac{bd}{A} \\ \frac{ae}{C} & \frac{bd}{A} & \frac{e^2}{C} \end{pmatrix}.$$

Symmetry Realization

- Under Z_{12} the leptonic fields transform as

$$\bar{l}_{R1} \to \omega \bar{l}_{R1}, \quad \bar{\nu}_{R1} \to \omega \bar{\nu}_{R1}, \quad D_{L1} \to \omega D_{L1},$$
$$\bar{l}_{R2} \to \omega^2 \bar{l}_{R2}, \quad \bar{\nu}_{R2} \to \omega^2 \bar{\nu}_{R2}, \quad D_{L2} \to \omega^3 D_{L2},$$
$$\bar{l}_{R3} \to \omega^5 \bar{l}_{R3}, \quad \bar{\nu}_{R3} \to \omega^5 \bar{\nu}_{R3}, \quad D_{L3} \to \omega^8 D_{L3},$$

- where $\omega = e^{i\pi/6}$
- The bilinears $\bar{l}_{Ra} D_{Lb}$ and $\bar{\nu}_{Ra} D_{Lb}$, relevant for $(M_l)_{ab}$ and $(M_D)_{ab}$ transform as

Symmetry Realization

$$\begin{pmatrix} \omega^2 & \omega^4 & \omega^9 \\ \omega^3 & \omega^5 & \omega^{10} \\ \omega^6 & \omega^8 & \omega \end{pmatrix}$$

while the bilinears $\bar{v}_{Ra} C \bar{v}_{Rb}^T$ relevant for $(M_R)_{ab}$ transform as

$$\begin{pmatrix} \omega^2 & \omega^3 & \omega^6 \\ \omega^3 & \omega^4 & \omega^7 \\ \omega^6 & \omega^7 & \omega^{10} \end{pmatrix}$$

Symmetry Realization

- To obtain a diagonal charged lepton mass matrix only three Higgs doublets are needed which transform under Z_{12} as
- $\Phi_{11} \longrightarrow \omega^{10}\Phi_{11}$
- $\Phi_{22} \longrightarrow \omega^{7}\Phi_{22}$
- $\Phi_{33} \longrightarrow \omega^{11}\Phi_{33}$

so the charged lepton mass matrix remains invariant. The non diagonal entries remain zero in the absence of Higgs doublets.

- Non zero entries of M_D and M_R can be obtained by introducing scalar Higgs doublets and scalar singlet fields
- $\Phi'_{13} \longrightarrow \omega^3 \Phi'_{13}$
- $\Phi'_{21} \longrightarrow \omega^9 \Phi'_{21}$
- $\Phi'_{22} \longrightarrow \omega^7 \Phi'_{22}$
- $\Phi'_{32} \longrightarrow \omega^4 \Phi'_{32}$
- $\Phi'_{33} \longrightarrow \omega^{11} \Phi'_{33}$

- $X_{12} \longrightarrow \omega^9 X_{12}$
- $X_{22} \longrightarrow \omega^8 X_{22}$
- $X_{33} \longrightarrow \omega^2 X_{33}$

- Under Z_2 the Φ'_{ab} and the neutrino singlets ν_{Ra} change sign while all other multiplets remain invariant

Weak Basis Invariants

Low energy CP-violation in the leptonic sector can be described using the following CP-odd WB invariants

$$I_1 = Img\ Det[H_\nu, H_l],$$

$$I_2 = Img\ Tr[H_l M_\nu M_\nu^* M_\nu H_l^* M_\nu^*],$$

$$I_3 = Img\ Det[M_\nu^* H_l M_\nu, H_l^*].$$

where $H_l = M_l^\dagger M_l$ *and* $H_\nu = M_\nu^\dagger M_\nu$

WBI from the neutrino mass matrix

In the earlier analysis it was found that there are correlations between Dirac and Majorana type CP violating phases. It is important to examine the interrelationships between CP odd weak basis invariants.The invariant I_1 was proposed by Jarlskog

C.Jarlskog, Phys.Rev.Lett. 55, 1039 (1985)

as a rephasing invariant measure of the Dirac type CP-violation in the quark sector. It, also, describes the CP-violation in the leptonic sector and is sensitive to the Dirac type CP-violating phase.

Invariants

The invariants I_2 and I_3 were proposed by Branco, Lavoura and Rebelo

G.C.Branco, L. Lavoura and M.N. Rebelo, Phys. Lett. B 180 (1986)264.

as the WB invariant measures of the Majorana type CP-violation.

- The CP violation in the lepton number conserving (LNC) processes is contained in Jarlskog CP invariant J which can be calculated from the WB invariant I_1 using the relation

$$I_1 = -2J\left(m_e^2 - m_\mu^2\right)\left(m_\mu^2 - m_\tau^2\right)\left(m_\tau^2 - m_e^2\right) \times \left(m_1^2 - m_2^2\right)\left(m_2^2 - m_3^2\right)\left(m_3^2 - m_1^2\right)$$

Where m_e, m_μ and m_τ are charged leptons masses and m_1, m_2 and m_3 are the eigenvalues of the complex neutrino mass matrix M_v.

In terms of the charged leptons masses and elements of M_v the invariant I_1 can be written as

$$I_1 = 2(m_e^2 - m_\mu^2)(m_\mu^2 - m_\tau^2)(m_\tau^2 - m_e^2) Img\ (M_{ee}A_{ee} + M_{\mu\mu}A_{\mu\mu} + M_{\tau\tau}A_{\tau\tau})$$

where the coefficients A_{ee}, $A_{\mu\mu}$ and $A_{\tau\tau}$ are given by

$$A_{ee} = M_{\mu\tau}M^{*}_{e\mu}M^{*}_{e\tau}(|M_{\mu\mu}|^2 - |M_{\tau\tau}|^2 - |M_{e\mu}|^2 + |M_{e\tau}|^2) + M_{\mu\mu}M^{*2}_{e\mu}(|M_{e\tau}|^2 - |M_{\mu\tau}|^2) + M^{*}_{\mu\mu}M^{*2}_{e\tau}M^{2}_{\mu\tau},$$

$$A_{\mu\mu} = M_{e\tau}M^{*}_{\mu\tau}M^{*}_{e\mu}(|M_{\tau\tau}|^2 - |M_{ee}|^2 - |M_{\mu\tau}|^2 + |M_{e\mu}|^2) + M_{\tau\tau}M^{*2}_{\mu\tau}(|M_{e\mu}|^2 - |M_{e\tau}|^2) + M^{*}_{\tau\tau}M^{*2}_{e\mu}M^{2}_{e\tau}$$

$$A_{\tau\tau} = M_{e\mu}M^{*}_{e\tau}M^{*}_{\mu\tau}(|M_{ee}|^2 - |M_{\mu\mu}|^2 - |M_{e\tau}|^2 + |M_{\mu\tau}|^2) + M_{ee}M^{*2}_{e\tau}(|M_{\mu\tau}|^2 - |M_{e\mu}|^2) + M^{*}_{ee}M^{*2}_{\mu\tau}M^{2}_{e\mu}.$$

S. Dev, Sanjeev Kumar and Surender Verma, Phys. Rev. D, 79 033011 (2009)

- The CP violation in lepton number violating (LNV) processes can be calculated from the WB invariants I_2 and I_3

$$I_2 = Img\,\big(M_{ee}M^{*2}_{e\mu}M_{\mu\mu}(m_e^2 - m_\mu^2)^2 + M_{\mu\mu}M^{*2}_{\mu\tau}M_{\tau\tau}(m_\mu^2 - m_\tau^2)^2 + M_{\tau\tau}M^{*2}_{e\tau}M_{ee}(m_\tau^2 - m_e^2)^2$$

$$+2M_{ee}M^{*}_{e\mu}M^{*}_{e\tau}M_{\mu\tau}(m_e^2 - m_\mu^2)(m_e^2 - m_\tau^2)$$

$$+2M_{\mu\mu}M^{*}_{\mu\tau}M^{*}_{e\mu}M_{e\tau}(m_\mu^2 - m_\tau^2)(m_\mu^2 - m_e^2)$$

$$+2M_{\tau\tau}M^{*}_{e\tau}M^{*}_{\mu\tau}M_{e\mu}(m_\tau^2 - m_e^2)(m_\tau^2 - m_\mu^2)\big)$$

$$I_3 = 2(m_e^2 - m_\mu^2)(m_\mu^2 - m_\tau^2)(m_\tau^2 - m_e^2)\,Img(m_e^2 M_{ee}B_{ee} + m_\mu^2 M_{\mu\mu}B_{\mu\mu} + m_\tau^2 M_{\tau\tau}B_{\tau\tau})$$

Where the coefficients B_{ee}, $B_{\mu\mu}$ and $B_{\tau\tau}$ are given by

$$B_{ee} = M_{\mu\tau}M^*_{e\mu}M^*_{e\tau}(m^4_\mu|M_{\mu\mu}|^2 - m^4_\tau|M_{\tau\tau}|^2 - m^2_e m^2_\mu|M_{e\mu}|^2 + m^2_e m^2_\tau|M_{e\tau}|^2) + m^2_\mu M_{\mu\mu}M^{*2}_{e\mu}(m^2_e|M_{e\tau}|^2 - m^2_\mu|M_{\mu\tau}|^2) + m^2_\mu m^2_\tau M^*_{\mu\mu}M^{*2}_{e\tau}M^2_{\mu\tau},$$

$$B_{\mu\mu} = M_{e\tau}M^*_{\mu\tau}M^*_{e\mu}(m^4_\tau|M_{\tau\tau}|^2 - m^4_e|M_{ee}|^2 - m^2_\mu m^2_\tau|M_{\mu\tau}|^2 + m^2_e m^2_\tau|M_{e\mu}|^2) + m^2_\tau M_{\tau\tau}M^{*2}_{\mu\tau}(m^2_\mu|M_{e\mu}|^2 - m^2_\tau|M_{e\tau}|^2) + m^2_e m^2_\tau M^*_{\tau\tau}M^{*2}_{e\mu}M^2_{e\tau}$$

$$B_{\tau\tau} = M_{e\mu}M^*_{e\tau}M^*_{\mu\tau}(m^4_e|M_{ee}|^2 - m^4_\mu|M_{\mu\mu}|^2 - m^2_e m^2_\tau|M_{e\tau}|^2 + m^2_\mu m^2_\tau|M_{\mu\tau}|^2) + m^2_e M_{ee}M^{*2}_{e\tau}(m^2_\tau|M_{\mu\tau}|^2 - m^2_e|M_{e\mu}|^2) + m^2_e m^2_\mu M^*_{ee}M^{*2}_{\mu\tau}M^2_{e\mu}.$$

Implications for texture zero and a vanishing minor

- For class 2A the invariants I_1, I_2 and I_3 are given by

$$I_1 = x\left(|M_{\mu\mu}|^2 + |M_{\mu\tau}|^2\right)\mathrm{Img}\left(M_{ee}M_{\tau\tau}M_{e\tau}^{*2}\right),$$

$$I_2 = \left(m_\tau^2 - m_e^2\right)^2\mathrm{Img}\left(M_{ee}M_{\tau\tau}M_{e\tau}^{*2}\right),$$

$$I_3 = x m_e^2 m_\tau^2\left(m_\mu^2|M_{\mu\mu}|^2 + m_\tau^2|M_{\mu\tau}|^2\right)\mathrm{Img}\left(M_{ee}M_{\tau\tau}M_{e\tau}^{*2}\right),$$

$$x = -2\left(m_e^2 - m_\mu^2\right)\left(m_\mu^2 - m_\tau^2\right)\left(m_\tau^2 - m_e^2\right).$$

The invariants for class 3A are obtained by interchanging μ and τ indices in the above relations

A necessary and sufficient condition for the absence of CP-violation for class 2A is

$$2\arg(M_{e\tau}) = \arg(M_{ee}) + \arg(M_{\tau\tau})$$

For class 2D the invariants I_1, I_2 and I_3 are given as

$$I_1 = -x\left(1 + \frac{|M_{e\tau}|^2}{|M_{ee}|^2}\right)\mathrm{Img}\left(M_{\mu\mu}M_{ee}^{*}M_{\mu\tau}^{*2}M_{e\tau}^{2}\right),$$

$$I_2 = \frac{\left(m_\mu^2 - m_\tau^2\right)^2}{|M_{ee}|^2}\mathrm{Img}\left(M_{\mu\mu}M_{ee}^{*}M_{\mu\tau}^{*2}M_{e\tau}^{2}\right),$$

$$I_3 = -x\left(m_e^2 m_\mu^2 m_\tau^2 + m_\mu^2 m_\tau^4 \frac{|M_{e\tau}|^2}{|M_{ee}|^2}\right)\mathrm{Img}\left(M_{\mu\mu}M_{ee}^{*}M_{\mu\tau}^{*2}M_{e\tau}^{2}\right).$$

The necessary and sufficient condition for CP Invariance for this class is

$$\arg(M_{ee}) + 2\arg(M_{\mu\tau}) = \arg(M_{\mu\mu}) + 2\arg(M_{e\tau})$$

The invariants for class 3F are obtained by interchanging μ and τ indices in the above relations.

S. Dev, Shivani Gupta and Radha Raman Guatam, J. Phys.: Nucl. Part. Phys. G 37, 125003 (2010), arXiv: hep-ph/1010.3839

For class 4B the invariants I_1, I_2 and I_3 are given by

$$I_1 = -x\left(1 + \frac{|M_{e\mu}|^2}{|M_{\mu\tau}|^2}\right) \mathrm{Img}\left(M_{ee} M_{\tau\tau}^{*} M_{e\mu}^{*2} M_{\mu\tau}^{2}\right),$$

$$I_2 = \left(\frac{|M_{e\tau}|^2 (m_\tau^2 - m_e^2)^2}{|M_{e\mu}|^2 |M_{\mu\tau}|^2} + \frac{2(m_e^2 - m_\mu^2)(m_e^2 - m_\tau^2)}{|M_{\mu\tau}|^2}\right) \mathrm{Img}\left(M_{ee} M_{\tau\tau}^{*} M_{e\mu}^{*2} M_{\mu\tau}^{2}\right),$$

$$I_3 = -x\left(m_e^2 m_\mu^2 m_\tau^2 + m_\mu^2 m_e^4 \frac{|M_{e\mu}|^2}{|M_{\mu\tau}|^2}\right) \mathrm{Img}\left(M_{ee} M_{\tau\tau}^{*} M_{e\mu}^{*2} M_{\mu\tau}^{2}\right).$$

Class 4B will be CP invariant if the phases of the neutrino mass matrix are fine tuned to satisfy the relation

$$\arg(M_{ee}) + 2\arg(M_{\mu\tau}) = \arg(M_{\tau\tau}) + 2\arg(M_{e\mu})$$

The three invariants are related to each other for all viable classes of neutrino mass matrices

Class	$\frac{I_1}{I_2}$	$\frac{I_1}{I_3}$
2A	$\frac{2(m_e^2-m_\mu^2)(m_\mu^2-m_\tau^2)(\lvert M_{\mu\mu}\rvert^2+\lvert M_{\mu\tau}\rvert^2)}{(m_e^2-m_\tau^2)}$	$\frac{(\lvert M_{\mu\mu}\rvert^2+\lvert M_{\mu\tau}\rvert^2)}{(m_e^2 m_\mu^2 m_\tau^2\lvert M_{\mu\mu}\rvert^2+m_e^2 m_\tau^4\lvert M_{\mu\tau}\rvert^2)}$
3A	$\frac{2(m_e^2-m_\tau^2)(m_\mu^2-m_\tau^2)(\lvert M_{\tau\tau}\rvert^2+\lvert M_{\mu\tau}\rvert^2)}{(m_\mu^2-m_e^2)}$	$\frac{(\lvert M_{\tau\tau}\rvert^2+\lvert M_{\mu\tau}\rvert^2)}{(m_e^2 m_\mu^2 m_\tau^2\lvert M_{\tau\tau}\rvert^2+m_e^2 m_\mu^4\lvert M_{\mu\tau}\rvert^2)}$
2D	$\frac{2(m_e^2-m_\mu^2)(m_\tau^2-m_e^2)(\lvert M_{ee}\rvert^2+\lvert M_{e\tau}\rvert^2)}{(m_\mu^2-m_\tau^2)}$	$\frac{(\lvert M_{ee}\rvert^2+\lvert M_{e\tau}\rvert^2)}{(m_e^2 m_\mu^2 m_\tau^2\lvert M_{ee}\rvert^2+m_\mu^2 m_\tau^4\lvert M_{e\tau}\rvert^2)}$
3F	$\frac{2(m_e^2-m_\mu^2)(m_\tau^2-m_e^2)(\lvert M_{ee}\rvert^2+\lvert M_{e\mu}\rvert^2)}{(m_\tau^2-m_\mu^2)}$	$\frac{(\lvert M_{ee}\rvert^2+\lvert M_{e\mu}\rvert^2)}{(m_e^2 m_\mu^2 m_\tau^2\lvert M_{ee}\rvert^2+m_\tau^2 m_\mu^4\lvert M_{e\mu}\rvert^2)}$
4B	$\frac{2(m_e^2-m_\mu^2)(m_\mu^2-m_\tau^2)(\lvert M_{e\mu}\rvert^2+\lvert M_{\mu\tau}\rvert^2)}{\left(\frac{\lvert M_{e\tau}\rvert^2(m_\tau^2-m_e^2)}{\lvert M_{e\mu}\rvert^2}+2(m_\mu^2-m_e^2)\right)}$	$\frac{(\lvert M_{e\mu}\rvert^2+\lvert M_{\mu\tau}\rvert^2)}{(m_e^2 m_\mu^2 m_\tau^2\lvert M_{\mu\tau}\rvert^2+m_\mu^2 m_e^4\lvert M_{e\mu}\rvert^2)}$
6C	$\frac{2(m_e^2-m_\tau^2)(m_\mu^2-m_\tau^2)(\lvert M_{e\tau}\rvert^2+\lvert M_{\mu\tau}\rvert^2)}{\left(\frac{\lvert M_{e\mu}\rvert^2(m_e^2-m_\mu^2)}{\lvert M_{e\tau}\rvert^2}+2(m_e^2-m_\tau^2)\right)}$	$\frac{(\lvert M_{e\tau}\rvert^2+\lvert M_{\mu\tau}\rvert^2)}{(m_e^2 m_\mu^2 m_\tau^2\lvert M_{\mu\tau}\rvert^2+m_\tau^2 m_e^4\lvert M_{e\tau}\rvert^2)}$

S. Dev, Shivani Gupta and Radha Raman Guatam, J. Phys.: Nucl. Part. Phys. G 37, 125003 (2010), arXiv: hep-ph/1010.3839

- These interrelationships of the invariants suggest that the three CP violating phases are not independent and there is only one independent physical phase in all phenomenologically viable neutrino mass matrices with texture zero and a vanishing minor.

THANK YOU