On the Relationship Between Translational Groups and Gauge Transformations

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Outline

- Wigner's little Group - A brief review

- Gauge generation by little group

  vector theories in 3+1 & 2+1 dimensions

  (Maxwell and Maxwell-Chern-Simons theories)

- Gauge generation in linearized gravity theories

  Linearized gravity in 3+1 d and Einstein-Chern-Simons theory in 2+1d

- Gauge generation in antisymmetric tensor theories

  Kalb-Ramond and $B \wedge F$ theories

- Conclusions

Wigner's Little Group

Definition: Subgroup of homogeneous Lorentz group that preserves the energy-momentum vector of a particle.

Little group for massive particles - Rotational groups

Little group for massless particle in 3+1 d:

$W_4(p,q;\phi) = W(p,q)R(\phi)$ preserves $k^a = (\omega, 0, 0, \omega)^T$.

$$W(p,q) \equiv W(p,q;0) = \begin{pmatrix} 1+\frac{p^2+q^2}{2} & p & q & -\frac{p^2+q^2}{2} \\ p & 1 & 0 & -p \\ q & 0 & 1 & -q \\ \frac{p^2+q^2}{2} & p & q & 1-\frac{p^2+q^2}{2} \end{pmatrix}$$

$W(p,q) \in T(2)$ - Translational group in 2 dimensions.

$p, \ q$ - continous parameters of translation.

$R(\phi)$ - Rotation about z axis $\in SO(2)$

Algebra isomorphic to $E(2)$:

Generators: $A = J_2 + K_1, \quad A = -J_1 + K_2, \quad J_3$

J -Rotation generators, $\quad K$ -Boost generators.

$$[A,B] = 0, \quad [J_3, A] = iB, \quad [J_3, B] = -iA.$$

Little group for *massless* particles in 2+1 d:

$$W(p) = \begin{pmatrix} 1+\frac{p^2}{2} & p & -\frac{p^2}{2} \\ p & 1 & -p \\ \frac{p^2}{2} & p & 1-\frac{p^2}{2} \end{pmatrix} \in \ T(1)$$

preserves $k^\mu = (\omega, 0, \omega)^T$.

Little group as a gauge generator:
Maxwell theory in 3+1 dimensions

[S.Weinberg, PRD(1964), YS Kim et.al PRD (1981,1982)]

$\mathcal{L} = -\frac{1}{4}F_{ab}F^{ab}; \qquad F_{ab} = \partial_a A_b - \partial_b A_a$

G.T $A_a(x) \to A_a(x) + \partial_a \Lambda(x)$, $\qquad \Lambda(x)$ -arbitrary

Equation of motion: $\partial_a F^{ab} = 0$.

Plane wave ansatz: $A^a(x) = \varepsilon^a(k)e^{ik\cdot x}$. $\Lambda(x) = \lambda(k)e^{ik\cdot x}$

Then in momentum space:
G.T.: $\varepsilon_a(k) \to \varepsilon_a(k) + i\lambda(k)k_a$.
Eqn. of Motion: $k^2\varepsilon^a - k^a k_b \varepsilon^b = 0$

Two possibilities-
(1) If $k^2 \neq 0$, then $\varepsilon^a \propto k^a \longrightarrow$ gauge artefact.
(2)If $k^2 = 0$, $k_a\varepsilon^a = 0 \longrightarrow \varepsilon^a(k) = (\varepsilon^0, \varepsilon^1, \varepsilon^2, \varepsilon^0)$ for a photon with $k^a = (\omega, 0, 0, \omega)^T$.

Maximally reduced form: $\qquad \varepsilon^a(k) = (0, \varepsilon^1, \varepsilon^2, 0)^T$

-displays just the two transverse physical degrees of freedom.

Action of $W(p,q)$:

$$\varepsilon^a \to \varepsilon'^a = W^a{}_b(p,q)\varepsilon^b = \varepsilon^a + \left(\frac{p\varepsilon^1 + q\varepsilon^2}{\omega}\right)k^a.$$

- gauge transformation if $\lambda(k) = \frac{p\varepsilon^1 + q\varepsilon^2}{i\omega}$.

Defining representation of translational subgroup $T(2)$ of Wigner's little group for massless particle acts as gauge generator in Maxwell theory.

By similar methods one can see that the same representation of the translational subgroup $T(2)$ of Wigner's little group for massless particle generates gauge tranformations also in

- Linearized gravity
  and

- Massless Kalb-Ramond theory.

2+1 dimensions:
Analogous to the gauge generation by $T(2)$ in 3+1 dimensional Maxwell theory, the translational group $W(p) \in T(1)$ generates gauge transformations in 2+1 dimensional Maxwell theory.

In this context one may also consider the gauge transformations in Maxwell-Chern-Simons theory which is a topologically massive gauge theory where gauge invariance co-exists with mass.

Maxwell-Chern-Simons (MCS) Theory (2+1 d)

$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \frac{\vartheta}{2}\epsilon^{\mu\nu\lambda}A_\mu\partial_\nu A_\lambda$

(topological) mass $|\vartheta|$, spin $= \frac{\vartheta}{|\vartheta|}$, ($\vartheta$ - real)

Eq. of motion: $\left(\Box g^{\mu\nu} + \vartheta\epsilon^{\mu\lambda\nu}\partial_\lambda\right)A_\nu = 0.$

Plane wave solution: $A^\mu(x) = \xi^\mu(k)\exp(ik.x)$.

Eq. of motion in terms of pol. vector $\xi^\mu(k)$:

$\xi_\nu k^\nu k^\mu - k^2\xi^\mu + i\vartheta\epsilon^{\mu\nu\lambda}\xi_\lambda k_\nu = 0.$

Here, $k^2 = 0$ (massless case) $\longrightarrow$ gauge artefacts.
Physical solution: Single massive mode, $k^2 = \vartheta^2$

Maximally reduced polarization vector (in the rest frame):

$\xi^\mu(\mathbf{0}) = \frac{1}{\sqrt{2}}\left(0, 1, -i\frac{\vartheta}{|\vartheta|}\right).$

ξ^μ have complex entries

$U(1)$ invariance - if ξ^μ are solutions, then $e^{i\phi}\xi^\mu$ are also solutions.

$W(p)$ fails to generate gauge transformations.

Construct new representations of the (2+1 d) little group:

$$D_\pm(p_\pm) = \begin{pmatrix} 1 & \pm p_\pm & \pm ip_\pm \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in T(1); \quad \text{isomorphic to } W(p)$$

$D_\pm(p_\pm)$ Generate gauge transformations in a doublet of MCS theories with opposite helicities.

$\xi^\mu_\pm \rightarrow D_\pm(p_\pm)\xi^\mu_\pm = \xi^\mu_\pm + \frac{\sqrt{2}p_\pm}{|\vartheta|}k^\mu$

[R. Banerjee, B.Chakraborty, TS; Mod. Phys. Lett. A (2001)]

Linearized gravity: (3+1 dimensions)

Gravity action: $I^E = -\int d^4x \mathcal{L}^E, \qquad \mathcal{L}^E = \sqrt{g}R$

R_{ab} - Ricci tensor

Linearized approximation: $g_{ab} = \eta_{ab} + h_{ab}$

η^{ab} flat metric,

h_{ab} - small deviation (only first order terms relevant).

Linearized Lagrangian: $\mathcal{L}_L^E = \frac{1}{2} h_{ab} \left[R_L^{ab} - \frac{1}{2}\eta^{ab} R_L \right].$

$$R_L^{ab} = \frac{1}{2}(-\Box h^{ab} + \partial^a \partial_c h^{cb} + \partial^b \partial_c h^{ca} - \partial^a \partial^b h)$$

linearized Ricci tensor.

Equation of motion: $R_L^{ab} - \frac{1}{2}\eta^{ab} R_L = 0$

Gauge invariance: $h^{ab} \to h^{ab} + \partial^a \Lambda^b(x) + \partial^b \Lambda^a(x)$

Plane wave ansatz: $h^{ab} = \chi^{ab} e^{ik.x}$

- Massive solutions are gauge arefacts
- Massless physical modes (with $k^a = (\omega, 0, 0, \omega)^T$)

  $$\longrightarrow \{\chi^{ab}\} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \chi^{11} & \chi^{12} & 0 \\ 0 & \chi^{12} & -\chi^{11} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

  maximally reduced polarization tensor where χ^{11}, χ^{12} are the independent elements representing the two physical degrees of freedom.

Gauge generation by $W(p,q)$

$$\{\chi^{ab}\} \to \{\chi'^{ab}\} = W(p,q)\{\chi^{ab}\}W^T(p,q) = \{\chi^{ab}\}$$

$$+\begin{pmatrix} A & B & C & A \\ A & 0 & 0 & A \\ B & 0 & 0 & B \\ A & B & C & A \end{pmatrix}.$$

$$A = (p^2 - q^2)\chi^{11} + 2pq\chi^{12}$$

$$B = p\chi^{11} + q\chi^{12}$$

$$B = p\chi^{12} - q\chi^{11}$$

This is a gauge transformation $\chi^{ab}(k) \to \chi^{ab}(k) + k^a\lambda^b(k) + k^b\lambda^a(k)$ if

$$\lambda^0 = \lambda^3 = \frac{A}{\omega}, \qquad \lambda^1 = \frac{B}{\omega}, \qquad \lambda^2 = \frac{C}{\omega}.$$

The Gauge transformation generated by $W(p,q)$ is only partial.

For the most general gauge transformation, all the components of the arbitrary vector field $\Lambda^a (= \lambda^a e^{ik.x})$ should be independent.

But $\lambda^0 = \lambda^3$. Therefore, tranlational group $T(2)$ generates only the gauge transformation obeying $k.\lambda = 0$ This is because $W(p,q)$ has only two parameters whereas Λ^a is a 4-component object.

[TS, B. Chakraborty, Class.Quant.Grav.(2002); TS, PRD(2003)]

Linearized Einstein-Chern-Simons (ECS) theory (2+1dimensions)

Pure gravity is a null theory in 2+1 d. Coupling to a Chern-Simons term provides a single massive physical mode.
Linearized version: (analogue of MCS theory)

$$\mathcal{L}_L^{ECS} = \mathcal{L}_L^{E} + \mathcal{L}_L^{CS}$$

$$= -\frac{1}{2}h_{\mu\nu}\left[R_L^{\mu\nu} - \frac{1}{2}\eta^{\mu\nu}R_L\right] - \frac{1}{2\mu}\epsilon_{\alpha\beta\gamma}\left[R_L^{\beta\delta} - \frac{1}{2}\eta^{\beta\delta}R_L\right]\partial^\alpha h_\delta^\gamma$$

Gauge invariance : $h_{\mu\nu} \to h_{\mu\nu} + \partial_\mu\Lambda_\nu(x) + \partial_\nu\Lambda_\mu(x)$.

For a pair of ECS theories (with opposite helicities):

- Massless solutions can be gauged away.
- Each has one physical massive degree of freedom; $k^2 = \mu^2$.
- Maximally reduced polarization tensors (rest-frame):
$\chi_\pm = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & \mp i \\ 0 & \mp i & -1 \end{pmatrix}$; $\qquad tr\left((\chi_\pm)^\dagger(\chi_\pm)\right) = 1$.

Note that $\chi_\pm^{\mu\nu} = \xi_\pm^\mu \xi_\pm^\nu$ (valid in any frame)
Pol. tensor of ECS theory is the direct product of the pol. vector of an MCS theory with itself.

$D_\pm(p_\pm) \in T(1)$ generates gauge transformations (partial)

$$\chi_\pm \to D_\pm(p_\pm)\chi_\pm D_\pm^T(p_\pm) = \chi_\pm + \begin{pmatrix} 2p_\pm^2 & p_\pm & \mp ip_\pm \\ p_\pm & 0 & 0 \\ \mp ip_\pm & 0 & 0 \end{pmatrix}.$$

with the choice $\zeta_0 = \frac{p_\pm^2}{|\mu|}, \quad \zeta_1 = \frac{p_\pm}{|\mu|}, \quad \zeta_2 = \frac{\mp ip_\pm}{|\mu|}$.

Massless Kalb-Ramond Theory:
(3+1 Dimensions)

$\mathcal{L} = \frac{1}{12} H_{abc} H^{abc}$; $\qquad H_{abc} = \partial_a B_{bc} + \partial_b B_{ca} + \partial_c B_{ab}$

B_{ab} is a 2-form gauge field: $B_{ab} = -B_{ba}$

Gauge symmetry: $B_{ab}(x) \to B_{ab}(x) + \partial_a F_b(x) - \partial_b F_a(x)$

-invariant under the transformation $F_a(x) \to F_a(x) + \partial_a \beta(x)$

Hence the gauge transformations are reducible.

Physical Excitations are massless: $k^2 = 0$

Maximally reduced polarization tensor:

$$\{\varepsilon^{ab}\} = \varepsilon^{12} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad \text{for } k^a = (\omega, 0, 0, \omega)^T$$

Gauge generation by $W(p,q)$: $\{\varepsilon^{ab}\} \to$

$$W(p,q)\{\varepsilon^{ab}\}W^T(p,q) = \{\varepsilon^{ab}\} + \varepsilon^{12} \begin{pmatrix} 0 & -q & p & 0 \\ q & 0 & 0 & q \\ -p & 0 & 0 & -p \\ 0 & -q & p & 0 \end{pmatrix}$$

This is a gauge transformation if

$f^1 = \frac{-q\varepsilon^{12}}{i\omega}, \qquad f^2 = \frac{p\varepsilon^{12}}{i\omega}, \qquad f^3 = f^0.$

Partial gauge generation- since only f^1 and f^2 are related to group parameters p, q.

Reducibility is manifest- since $f^3 = f^0$ and are independent of p, q

$B \wedge F$ theory (3+1 d)
Topologically massive gauge theory

$$\mathcal{L} = -\frac{1}{4}F_{ab}F^{ab} + \frac{1}{12}H_{abc}H^{abc} - \frac{m}{6}\epsilon^{abcd}H_{abc}A_d$$

Physical excitations are massive; $k^2 = m^2$.

Maximally reduced pol. vector and tensor:

$$\varepsilon^a = -i\begin{pmatrix} 0 \\ a \\ b \\ c \end{pmatrix};\ \{\varepsilon^{ab}\} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & c & -b \\ 0 & -c & 0 & a \\ 0 & b & -a & 0 \end{pmatrix};\ \varepsilon^{ab}\varepsilon_b = 0.$$

Question: What generates gauge transformations of $B \wedge F$ theory?

Consider $D(p,q,r) = \begin{pmatrix} 1 & p & q & r \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$.

Then, $D^a{}_b(p,q,r)\varepsilon^b = \varepsilon^a - \frac{i}{m}(pa+qb+rc)k^\mu$ and

$$D(p,q,r)\{\varepsilon^{ab}\}D^T(p,q,r) = \{\varepsilon^{ab}\} + \begin{pmatrix} 0 & (rb-qc) & (pc-ra) & (qa-pb) \\ -(rb-qc) & 0 & 0 & 0 \\ -(pc-ra) & 0 & 0 & 0 \\ -(qa-pb) & 0 & 0 & 0 \end{pmatrix}$$

These constitute the full set of gauge transformations:

$\lambda(k) = \frac{pa+qb+rc}{im}$ and

$\lambda^1(k) = \frac{rb-qc}{im}, \quad \lambda^2(k) = \frac{pc-ra}{im}, \quad \lambda^3(k) = \frac{qa-pb}{im}.$

$\lambda^0(k)$ - undetermined $\longrightarrow$ reducibilty.

Note, $D(p,q,r)\cdot D(p',q',r') = D(p+p',q+q',r+r')$

(multiplication rule for $T(3)$, 3-d translational group)

Also generators of $D(p,q,r)$ satisfy $T(3)$ algebra.

Translation group $T(3)$ in the representation $D(p,q,r)$ generates gauge transformations in $B \wedge F$ theory.

[R. Banerjee, B. Chakraborty, Phys. Lett. B(2001)]

Conclusions

- Suitable representations of the translational subgroup of Wigner's little group for massless particles generate gauge transformations different abelian gauge theories.

- In the case of reducible gauge systems, the gauge generation by translational group manifests this reducibility.

- In certain cases, the gauge generation by translational group covers only a subset of the full spectrum of gauge transformations.