Equivalency between different approaches of finding the positive definite metric in pseudo-Hermitian theories

Ananya Ghatak , Bhabani Prasad Mandal

Department of Physics, Banaras Hindu University, VARANASI

Email: gananya04@gmail.com

Plan of the Talk

- Pseudo-Hermitian Hamiltonian: Definitions
- Motivations for such studies
- Problems with Quantum Mechanical study
- Equivalence of metric operators
- Conclusions

Definitions:

A Hamiltonian is Hermitian in the sense of Dirac if it satisfies,

$$H=H^{\dagger}$$

A *pseudo-Hermitian Hamiltonian* is not Dirac Hermitian rather it satisfies

$$H=S^{-1}H^{\dagger}S\,,\quad S=S^{\dagger}$$

If **S=I** then H becomes Dirac Hermitian

Motivations

- Bose hard sphere : T T Wu in 1950s
- Lee-Yang edge singularity: M. Fisher, J. Cardy in 1970s
- Complex crystal: M. Berry in 1990s
- Integrable systems
- PT-Symmetric : C. Bender et. al 1998s

Problems in probabilistic interpretation

- Eigenstates do not satisfy standard completeness relation.
- Norms are not positive definite, if we consider Dirac inner product.

$$\langle \varphi | \psi \rangle = \int \varphi^* (x) \psi(x) dx = \langle \psi | \varphi \rangle^*$$

- Hence in such theories we face problems with probabilistic description as well as unitary time evolution!!

Solutions

- In pseudo-Hermitian theory we have additional operator S, which allow us to define a modified inner product,

$$\langle \varphi | \psi \rangle_S = \langle \varphi | S\psi \rangle = \langle S\varphi | \psi \rangle$$

- The adjoint with respect to this inner product can be defined as,

$$\langle H^{\#}\varphi | \psi \rangle_S = \langle \varphi | H\psi \rangle_S = \langle \varphi | SH\psi \rangle = \langle H^{\dagger} S\varphi | \psi \rangle$$
$$= \langle SS^{-1} H^{\dagger} S\varphi | \psi \rangle = \langle S^{-1} H^{\dagger} S\varphi | \psi \rangle_S$$

- Hence the adjoint with respect to the modified inner product,

$$H^{\#} = S^{-1} H^{\dagger} S$$

- Hamiltonian become self-adjoint with respect to this modified product.

$$H^{\#} = H$$

- As a result the time evolution is unitary,

$$\langle \varphi(t)|\psi(t)\rangle_S = \langle e^{-iHt}\varphi(o)|e^{-iHt}\psi(o)\rangle_S$$
$$= \langle e^{+iH^{\#}t}e^{-iHt}\varphi(o)|\psi(o)\rangle_S$$
$$= \langle \varphi(o)|\psi(o)\rangle_S$$

- However this product may not be positive ⟹ Probabilistic interpretation may have problem.

- We can have a consistent quantum theory for pseudo Hermitian systems if we can find a positive definite S to modify the inner product.

- There are many different approaches to construct such positive definite inner product.

1. **Spectral Method by Ali Mostafazadeh,**
2. **Method introduced by Ashok Das.**

Spectral Method by Ali Mostafazadeh

In the Spectral method the positive definite η for a pseudo Hermitian theory is defined as,

$$\eta = \Sigma |u_i\rangle\langle u_i|,$$

Where $|u_i\rangle$ is the eigen vectors corresponding to $H^\dagger$

$$H^\dagger |u_i\rangle = \lambda_i |u_i\rangle$$

Ashok Das Method

- If any operator A, commute with H, $[A,H]=0$, then we can define a new inner product w.r.to $q = SA$, $\langle\varphi|\psi\rangle_q = \langle\varphi|SA|\psi\rangle$,

Which will allow the probabilistic description while maintaining unitary time evolution if A is properly chosen.

- The adjoint w.r.to q

$$H^{\ddagger} = q^{-1}H^{\dagger}q = A^{-1}S^{-1}H^{\dagger}SA$$
$$= A^{-1}HA = H$$

- Next we define the Hilbert space of this theory as,

$$\mathcal{H} : \{|\psi_E\rangle ; 0 < |\langle\psi_{E^*}|\psi_E\rangle| < \infty$$

- If P_E denote the projection operator onto such state then,

$$P_E|\psi_E\rangle = \delta_{EE'}|\psi_{E'}\rangle$$

- This space is completed by construction, so any operator which commutes with H can be written as

$$A = \sum_E c_E P_E \, , A|\psi_E\rangle = c_E|\psi_E\rangle$$

- Thus

$$q = SA = S \sum_E c_E P_E$$

A Das et. al showed that it is always possible to chose a c_E such that norms are definite with q inner product.

Equivalence of inner products

We have shown by explicit calculations that the metric operators constructed in the two completely different approaches are equivalent.

Model I: System of spin ½ particle in magnetic field

The Hamiltonian of a system of a spin ½ particle in a magnetic field **B** coupled to a oscillator through some non-Hermitian interaction is,

$$H = \mu\sigma . B + \hbar\omega a^\dagger a + \rho(\sigma_+ a - \sigma\_a^\dagger)$$

for $B = B_0\hat{z}, H = {}^{\varepsilon}/_{2}\, \sigma_z + \hbar\omega a^\dagger a + \rho(\sigma_+ a - \sigma\_a^\dagger)$

In the matrix form for the states $|n, 1/2\rangle$ & $|n+1, -1/2\rangle$,

$$H_{n+1} = \begin{pmatrix} \frac{\varepsilon}{2} + n\hbar\omega & \rho\sqrt{n+1} \\ -\rho\sqrt{n+1} & -\frac{\varepsilon}{2} + (n+1)\hbar\omega \end{pmatrix}$$

The constructed inner product by the method by A Das and by the spectral method by M Ali is the following,

$$q = \begin{pmatrix} 1 & sin\theta_{n+1} \\ sin\theta_{n+1} & 1 \end{pmatrix}$$

Where,

$$2\rho\sqrt{n+1} = (\hbar\omega - \epsilon) sin\,\theta_{n+1}$$

Model II: An arbitrary two level system

Here we have the Hamiltonian[5] as,

$$H = \begin{pmatrix} re^{i\theta} & s \\ s & re^{-i\theta} \end{pmatrix}$$

By Ashok Das method the inner product is,

$$q = \left(\frac{1}{R}\right)\begin{pmatrix} s & -irsin\theta \\ irsin\theta & s \end{pmatrix}$$

(Where, $R = \sqrt{s^2 - r^2 sin^2\theta}$)

Which is the same as calculated by Ali's method.

Model III : Relavistic scalar interaction

For a Dirac particle of mass m_0 subjected to a Pseudo-Hermitain potential V_s the Hamiltonian be,

$$H = c\alpha . p + \beta m_0 c^{2} + V_S ,$$

where, $$V_S = V_0 \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

so $$H = \begin{bmatrix} m_0 c^2 & cp_x + V_0 \\ cp_x - V_0 & -m_0 c^2 \end{bmatrix}$$

In both above methods the inner-product is,

$$q = \frac{E + m_0 c^2}{2E} \begin{bmatrix} 1 + \frac{(cp_x - V_0)^2}{(E + m_0 c^2)^2} & \frac{2V_0}{E + m_0 c^2} \\ \frac{-2V_0}{E + m_0 c^2} & 1 + \frac{(cp_x + V_0)^2}{(E + m_0 c^2)^2} \end{bmatrix}$$

Conclusions

- Consistent quantum theory can be developed with pseudo-Hermitian Hamiltonian by modifying the inner product.

- There are several approaches to find the positive definite inner product.

- We have shown with several explicit examples that such metric operators are equivalent.

Thanks for your attention…