QUARK NUMBER SUSCEPTIBILITY IN HARD THERMAL LOOP PERTURBATION THEORY

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Quark Number Susceptibility

- Fluctuation of some quantities are important and related to the correlation functions.
- Fluctuations $\Longrightarrow$ Susceptibility $\Longrightarrow$ measures the response of a system from its equilibrium state to the external field.
- QNS is the measured of response of quark number density to infinitesimal change in quark chemical potential.

$$\chi_q(T) = \left.\frac{\partial \rho(T,\mu)}{\partial \mu}\right|_{\mu=0}$$

- QNS can also be expressed as current-current correlation function

$$\chi_q(T) = \beta \int d^3x \langle j_0(0,\vec{x}) j_0(0,\vec{0})\rangle = \beta \int d^3x S_{00}(0,\vec{x}).$$

Feynman Rules in finite temperature

- Propagator:
  Replace k_0 in the propagator by $i\omega_n$ where Matsubara frequency

$$\begin{aligned}\omega_n &= 2n\pi T && \text{for boson}\\ &= (2n+1)\pi T && \text{for fermion.}\end{aligned}$$

- Vertex: same as at $T = 0$;
- Loop integration:

$$\int \frac{d^4K}{(2\pi)^4} \longrightarrow \frac{i}{\beta} \sum_{n=-\infty}^{n=\infty} \int \frac{d^3k}{(2\pi)^3}.$$

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Problem of normal perturbation theory in finite temperature

1. Gauge dependent gluon damping rate.
2. Infrared singularities for mass $m \to 0$ though those diagrams are infrared finite at $T = 0$.

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- *Two scales of momentum at high temperature($T \gg m$):*
  1. **Hard Momentum** $\Rightarrow k_0$ **and** $k \sim T$**,** $K = (k_0, k)$
  2. **Soft Momentum** $\Rightarrow k_0, k \sim gT$

$\Longrightarrow$ *In HTL one has to distinguish between Hard and Soft momentum.*

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- *One has to resum the Hard Thermal Loop (HTL) in geometrical series in the form of effective propagators and vertices in one loop and related by the Ward Identity.*

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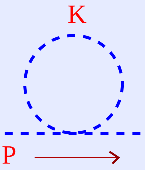
- *One has to resum the Hard Thermal Loop (HTL) in geometrical series in the form of effective propagators and vertices in one loop and related by the Ward Identity.*
- *These can be used in ordinary perturbation theory to compute the thermodynamic quantities.*

HTL in Scalar theory

The one loop self energy
for $\mathcal{L}_{int} = -g^2\phi^4$:



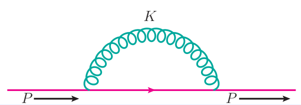
After subtraction the vacuum contribution it gives

$$\Pi_1 = -12g^2 \int \frac{d^4K}{(2\pi)^4} \frac{1}{K^2} = 12g^2 T \sum_{n=-\infty}^{n=\infty} \int \frac{d^3k}{(2\pi)^3} \frac{1}{\omega_n^2 + k^2} = g^2 T^2.$$

In Scalar theory, the ressumation is just a matter of replacing bare propagator $1/P^2$, by an effective one, $1/(P^2 + \Pi_1)$.
So effective propagator $= \frac{1}{P^2+g^2T^2}$.

HTL in Gauge theory

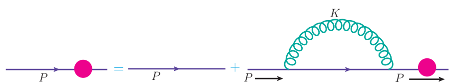
- Quark self energy in HTL



$$\Sigma_{HTL}(P) = \frac{m_q^2}{p}\gamma_0 Q_0\left(\frac{p_0}{p}\right) + \frac{m_q^2}{p}\left\{1 - \frac{p_0}{p}Q_0\left(\frac{p_0}{p}\right)\right\}\vec{\gamma}\cdot\hat{p}, \qquad m_q^2 = \frac{1}{6}g^2T^2.$$

where $Q_0\left(\frac{p_0}{p}\right) = \frac{1}{2}\ln\frac{p_0+p}{p_0-p}$ is Legendre function of second kind.

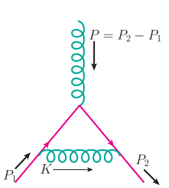
- The full propagator(Two point HTL function)



$$S^\star(P) = \frac{1}{\not{P} - \Sigma(P)} = -\frac{1}{2}\left[\frac{\gamma^0 - \vec{\gamma}\cdot\hat{p}}{D_+(P)} + \frac{\gamma^0 + \vec{\gamma}\cdot\hat{p}}{D_-(P)}\right] \quad \text{where}$$

$$D_\pm(p_0,p) = -p_0 \pm p + \frac{m_f^2}{p}\left[\pm 1 + \frac{1}{2}\left(1 \mp \frac{p_0}{p}\right)\ln\frac{p_0+p}{p_0-p}\right].$$

- The three point HTL function



$$\Gamma^{\mu}_{HTL}(P_1,P_2;P) = \gamma^\mu + m_q^2 \int \frac{d^3k}{(2\pi^3)} \frac{\hat{\not{K}}\hat{K}^\mu}{(\hat{K}\cdot P_1)(\hat{K}\cdot P_2)}$$
$$= (\gamma^\mu + \delta\Gamma^\mu)$$

- HTL two electron-two photon function

$$\Gamma^{\mu\nu}(P_1,P_2;Q_1,Q_2) = -m_f^2 \int \frac{d\Omega}{4\pi} \frac{\hat{\not{K}}\hat{K}^\mu\hat{K}^\nu}{\left[(P_1+Q_1)\cdot\hat{K}\right]\left(P_2-Q_1)\cdot\hat{K}\right]} \times \left[\frac{1}{P_1.\hat{K}} + \frac{1}{P_2.\hat{K}}\right].$$

HTL Lagrangian in Gauge theory

The HTL improved terms for gluon and quark.

$$\tilde{\mathcal{L}}_{HTL} = -\frac{3}{2} m_g^2 \mathrm{Tr} \left(F_{\mu\nu} \left\langle \frac{\hat{K}^\nu \hat{K}^\rho}{(\hat{K} \cdot D)^2} \right\rangle F_\rho^\mu \right) + m_q^2 \bar{\psi} \gamma_\mu \left\langle \frac{\hat{K}^\mu}{\hat{K} \cdot D} \right\rangle \psi$$

- 1st term is for gluon, $m_g =$ thermal gluon mass.
- Second term is for quark. $m_q =$ thermal quark mass.
- $\hat{K} = (1, \hat{k})$ is a light like four vector and D^μ is covariant derivative.
- $F_{\mu\nu}$ is gluon field tensor.
- ψ is fermionic field.
- $\langle\rangle$ is averaging over angular direction.
- Gauge independent.

The HTL Lagrangian for quark including HTL correction term:

$$\mathcal{L}_{HTL} = \mathcal{L}_{QCD} + \tilde{\mathcal{L}}_{HTL} = \bar{\psi} i\gamma^\mu D_\mu \psi + m_q^2 \bar{\psi}\gamma^\mu \left\langle \frac{\hat{K}_\mu}{i\hat{K}\cdot D} \right\rangle \psi \ ,$$

Now we assume D_μ as $(D_\mu - i\delta_{\mu 0} j)$ where j can be an external source or perturbation to which the response of the system can be calculated. Expanding in power of j

$$\mathcal{L}_{HTL} = \bar{\psi}\left(i\not{D} + m_q^2 \left\langle \frac{\hat{\not{K}}}{i\hat{K}\cdot D} \right\rangle \right)\psi + j\bar{\psi}\left(\delta^{\mu 0}\gamma_\mu - m_q^2 \left\langle \frac{\hat{\not{K}}\hat{K}_\mu \delta^{\mu 0}}{(i\hat{K}\cdot D)^2} \right\rangle \right)\psi$$
$$+ m_q^2 j^2 \bar{\psi} \left\langle \frac{\hat{\not{K}}\hat{K}_\mu \hat{K}_\nu \delta^{\mu 0}\delta^{\nu 0}}{(i\hat{K}\cdot D)^3} \right\rangle \psi + \mathcal{O}(j^3)$$

Using the Lagrangian and considering j as chemical potential μ, we can calculate partition function $\mathcal{Z}(T,\mu)$ and taking the derivative of $\ln \mathcal{Z}(T,\mu)$ quark number density becomes

$$\rho[\mu] = N_c N_f T \int \frac{d^3k}{(2\pi)^3} \sum_{k_0} \mathsf{Tr}\left[S^\star[\mu]\Gamma_0[\mu] + \frac{3}{2}\mu S^\star[\mu]\Gamma_{00}[\mu] + \mathcal{O}(\mu^2) \right]$$

The effective HTL propagator for momentum K is given as

$$S^\star(K) = -\frac{\gamma_0 - \vec{\gamma}\cdot\hat{k}}{2D_+(k_0,k)} - \frac{\gamma_0 + \vec{\gamma}\cdot\hat{k}}{2D_-(k_0,k)} \ ,$$

with

$$D_\pm(k_0,k) = -k_0 \pm k + \frac{m_q^2}{k}\left[\frac{1}{2}\left(1 \mp \frac{k_0}{k}\right)\ln\frac{k_0+k}{k_0-k} \pm 1\right],$$

$$m_q^2 = \frac{g^2}{6}\left(T^2 + \frac{\mu^2}{\pi^2}\right) \ , \quad g^2 = 4\pi\alpha_s = \frac{24\pi^2}{(33-2N_f)\ln\left(\frac{Q}{\Lambda_0}\right)}$$

At zero external momentum and small energy limit, the 3-point HTL function can be obtained from the Ward identity as

$$\Gamma^0(K,-K;0) = \frac{\partial}{\partial k_0}\left(S^\star(K)^{-1}\right) = a\gamma^0 + b\vec{\gamma}\cdot\hat{k},$$

where

$$a \pm b = -D'_\pm(k_0,k) = -\frac{D_\pm}{k_0 \mp k} + \frac{2m_q^2}{k_0^2 - k^2}$$

Thermodynamics in HTLpt

Number density

Thermodynamics in HTLpt

Number density

$$\rho^{HTL}(T,\mu) = N_c N_f \int \frac{d^4K}{(2\pi)^4}\, \mathsf{Tr}\left[S^\star[\mu]\Gamma_0[\mu]\right]$$

$$= 2N_c N_f T \int \frac{d^3k}{(2\pi)^3} \sum_{k_0} \left[\frac{1}{k_0 - k} + \frac{1}{k_0 + k} - \frac{2m_q^2}{k_0^2 - k^2}\left(\frac{1}{D_+} + \frac{1}{D_-}\right)\right] .$$

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$$\rho^{QP}(T,\mu) = 2N_c N_f \int \frac{d^3k}{(2\pi)^3} \left[n(\omega_+-\mu) + n(\omega_--\mu) - n(k-\mu) - n(\omega_++\mu) - n(\omega_-+\mu) + n(k+\mu)\right] \quad ,$$

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$$\rho^{QP}(T,\mu) = 2N_c N_f \int \frac{d^3k}{(2\pi)^3}\left[n(\omega_+-\mu)+n(\omega_--\mu)-n(k-\mu)\right.$$
$$\left. -n(\omega_++\mu)-n(\omega_-+\mu)+n(k+\mu)\right] ,$$

$$\rho^{LD}(T,\mu) = N_c N_f \int \frac{d^3k}{(2\pi)^3} \int_{-k}^{k} d\omega \left(\frac{2m_q^2}{\omega^2-k^2}\right) \beta_+(\omega,k)\left[n(\omega-\mu)-n(\omega+\mu)\right] .$$

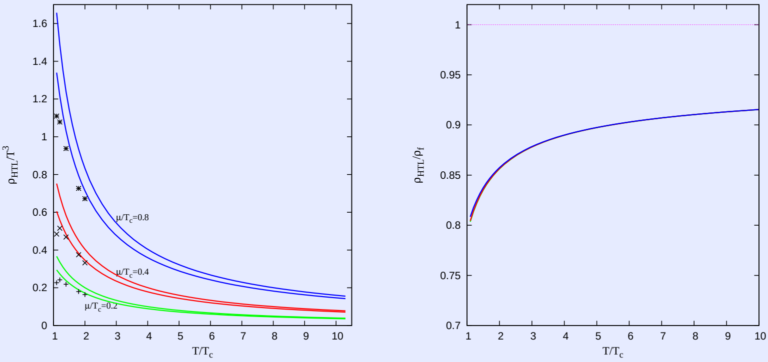

Figure: The choice of the renormalisation scale for both panels is $Q = 2\pi T$

Thermodynamics in HTLpt

Pressure

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Pressure in HTL approximation can be obtained by integrating number density as

$$\mathcal{P}^{HTL}(T,\mu) = \int_0^\mu d\mu' \ \rho(T,\mu').$$

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Entropy density

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$$\mathcal{P}^{HTL}(T,\mu) = \int_0^{\mu} d\mu' \; \rho(T,\mu').$$

Entropy density

Entropy density in HTL approximation can be obtained by differentiating HTL Pressure.

$$\mathcal{S}^{HTL}(T,\mu) = \frac{\partial}{\partial T}\left[\mathcal{P}^{HTL}(T,\mu)\right].$$

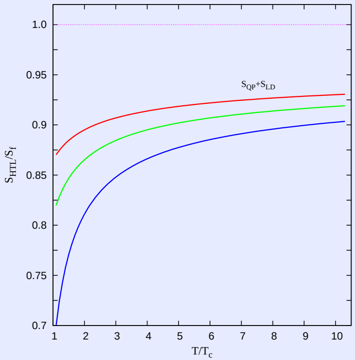

Figure: The ratio of HTL to free quark entropy as a function of T/T_C at $\mu/T_C = 0$ (left panel) and $\mu/T_C = 0.2$ (right panel). The different choices of the renormalisation scale are $Q = \pi T$ (Blue), $2\pi T$ (Green), and $4\pi T$ (Red).

Quark Number Susceptibility

$$\chi^{QP}(T) = \frac{\partial}{\partial\mu}\left[\rho^{QP}(T,\mu)\right]\Big|_{\mu=0}$$

$$= 4N_cN_f\beta\int\frac{d^3k}{(2\pi)^3}\left[n(\omega_+)\left(1-n(\omega_+)\right) + n(\omega_-)\left(1-n(\omega_-)\right) - n(k)\left(1-n(k)\right)\right]$$

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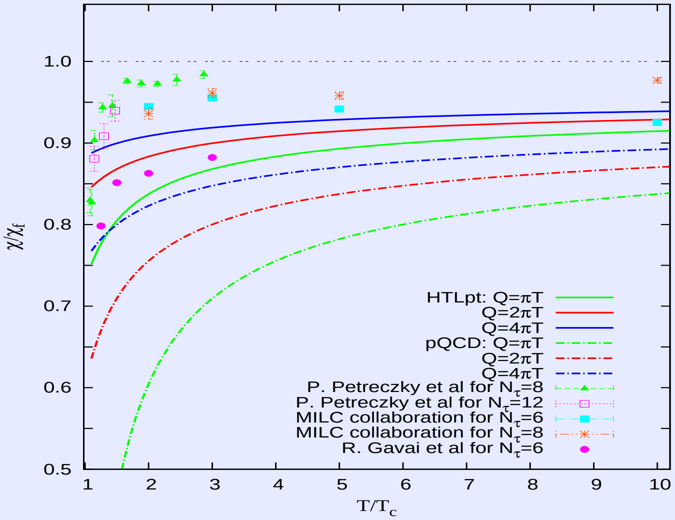
$$\chi^{LD}(T) = \frac{\partial}{\partial\mu}\left[\rho^{LD}(T,\mu)\right]\Big|_{\mu=0} = 2N_cN_f\beta\int\frac{d^3k}{(2\pi)^3}\int_{-k}^{k}d\omega\left(\frac{2m_q^2}{\omega^2-k^2}\right) \times \beta_+(\omega,k)\;n(\omega)\,(1-n(\omega))$$


1.0
0.9
0.8
0.7
0.6
0.5
1
2
3
4
5
6
7
8
9
10
χ/χ_f
T/T_c
HTLpt: Q=πT
Q=2πT
Q=4πT
pQCD: Q=πT
Q=2πT
Q=4πT
P. Petreczky et al for N_τ=8
P. Petreczky et al for N_τ=12
MILC collaboration for N_τ=6
MILC collaboration for N_τ=8
R. Gavai et al for N_τ=6

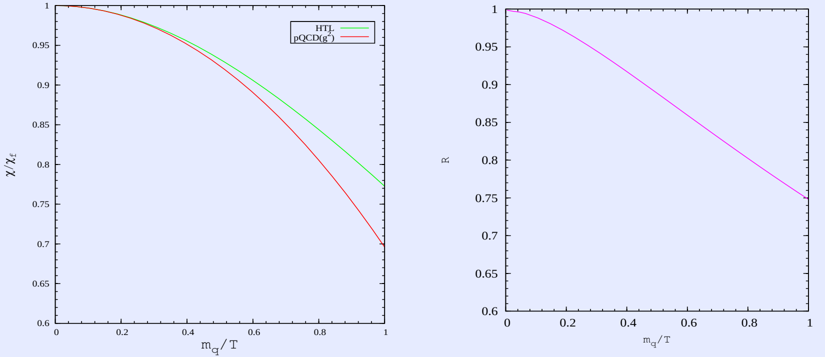


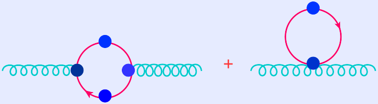
Figure: The ratio of HTL to free QNS as a function of m_q/T (left panel) and the ratio of deviation of HTL QNS from free one to the deviation of LO perturbative QNS from free on $\left[R = \frac{\chi^{HTL} - \chi^{free}}{\chi^{pert} - \chi^{free}}\right]$ as a function of m_q/T (right panel).

Thermodynamic sum rule and QNS

Here we intend to obtain QNS, associated with the conserved quantity, by first taking derivative w.r.t. μ of $\rho(\mu, T)$ and then perform the frequency sums. Number density

$$\rho[\mu] = iN_cN_fT\int\frac{d^3k}{(2\pi)^3}\sum_{k_0}\mathrm{Tr}\left[S^\star[\mu]\Gamma_0[\mu] + \mu S^\star[\mu]\Gamma_{00}[\mu] + \mathcal{O}(\mu^2)\right]$$

$$\chi(T) = N_cN_fT\int\frac{d^3k}{(2\pi)^3}\sum_{k_0}\mathrm{Tr}\left[S^\star(K)\Gamma_0(K,-K;0)S^\star(K)\Gamma_0(-K,K;0)\right.$$
$$\left.-2S^\star(K)\Gamma_{00}(K,-K;0,0)\right]$$

$=$ 

This equation is known as **thermodynamics sum rule** as it relates the thermodynamic derivative($\chi(T)$) to the correlation function due to symmetry of the system. By calculating frequency sum of RHS,

$$\chi^{HTL}(T) = \chi^{QP}(T) + \chi^{LD}(T)$$

Conclusion and Outlook

- Problems in BPT has been discussed.
- Improvement of the situation through HTL approximation and HTLpt has been discussed.
- HTLpt are used to calculate one loop QNS and also other thermodynamic quantities in QGP.
- Thermodynamic sum rule has been verified.

Reference: arXiv:1007.2076 [hep-ph] by Najmul Haque, Munshi G. Mustafa.

THANK YOU.