Calculation of chiral condensate in Lattice QCD with 2-flavour dynamical Wilson fermion

Santanu Mondal[1]

December 14, 2010

Collaborators: Asit K. De[1], Sangita De Sarkar[1], A. Harindranath[1], Jyotirmoy Maiti[2], Anwesa Sarkar[1]

[1]Theory Division, Saha Institute of Nuclear Physics, 1/AF Bidhannagar, Kolkata : 700064, India

[2]Department of Physics, Barasat Govt. College, 10 KNC road, Barasat, Kolkata: 700124, India

Why Chiral Condensate?

The fermionic part of Lagrangian density for QCD:

$$\mathscr{L}_{QCD} = i\overline{\psi}_L \not{D} \psi_L + i\overline{\psi}_R \not{D} \psi_R - \overline{\psi}_L M \psi_R - \overline{\psi}_R M^\dagger \psi_L$$

Chiral transformation:

$$\overline{\psi}_{L,R} \to \overline{\psi}_{L,R} U^\dagger_{L,R}, \quad \psi_{L,R} \to U_{L,R}\psi_{L,R} : \qquad U_{L,R} \in SU(3)_{L,R}.$$

$M = M^\dagger = diag(m_u, m_d, m_s)$: breaks chiral symmetry explicitly

$$m_u \ = \ 1.5 - 3.3 \ MeV, m_d \ = \ 3.5 - 6 \ MeV \quad << \quad \Lambda_{\text{QCD}} \ \approx \ 300 \text{ MeV}$$

$$m_s = 105^{+25}_{-35} \text{ MeV}$$

(Quark masses are extracted in $\overline{MS}$ scheme at a scale $\mu \approx 2GeV$)

So, $SU(2)_L \times SU(2)_R$ is a good approximate symmetry, and $SU(3)_L \times SU(3)_R$ is more approximate symmetry.

Approximate chiral symmetry $G = SU(3)_L \times SU(3)_R$ is spontaneously broken down to $H = SU(3)_v \rightarrow$ generates 8 Goldstone bosons $\rightarrow$ identified with π, K and η.
This is based on :

- The lightness of π, K and η are consistent with being pseudo Goldstone bosons of the broken approximate chiral symmetry in QCD.
- The remarkable success of ChPT specially in the $SU(2)$ sector.

Since vacuum is invariant under Lorentz transformation and under space reflections, only scalar operators $(\overline{\psi}\psi)$ can have nonzero vacuum expectation values. $\overline{\psi}\psi$ breaks axial G symmetry. In spontaneous breaking phase of $G \rightarrow H$ vacuum is not invariant under axial $G \Rightarrow \overline{\psi}\psi$ has nonzero VEV in this phase, where as $\langle\overline{\psi}\psi\rangle = 0$ in symmetry restored phase.
$\Rightarrow \langle\overline{\psi}\psi\rangle$ can act as an order parameter.

Chiral Condensate on Lattice

- Chiral condensate (with the Wilson-type fermions):
  - Not easy because the scalar density operator $\overline{\psi}\psi$ has leading power divergence of the form m_q/a^2 as the cutoff $1/a$ goes to infinity.
  - When the chiral symmetry is violated from the outset as in the Wilson-type fermions, the divergence is even stronger $1/a^3$
    $\Longrightarrow$ direct measurement of chiral condensate on lattice and then taking the chiral continuum limit will not give the desired condensate.
  - The condensate however vanishes in the massless limit, in finite volume
    $\Longrightarrow$proper order of the limits is to take the infinite volume limit first and then the massless limit.

Need some theoretical guidance; provided by ChPT.

ChPT Results

Considering only two degenerate flavours, up and down, in ChPT,
In LO:

$$\langle 0|\overline{u}u|0\rangle_0 = \langle 0|\overline{d}d|0\rangle_0 = -F_0^2 B \;\rightarrow\; \text{depends on two parameters}$$

In NLO:

$$\langle 0|\overline{u}u|0\rangle = \langle 0|\overline{d}d|0\rangle_0 = \langle 0|\overline{u}u|0\rangle_0[1 - \frac{3M^2}{32\pi^2 F_0^2}\ln \underbrace{\frac{M^2}{\Lambda_3^2}}_{\text{extra parameter in NLO}}] \,, \quad M^2 = 2Bm$$

Chiral condensate from Ward identity

The fermionic part of action in continuum for QCD in Euclidean space:

$$S = \int d^4x \mathscr{L}_{QCD} = \int d^4x [\overline{\psi} \not{D} \psi + \overline{\psi} M \psi]$$

Now consider axial transformation:

$$\delta\psi = i\alpha^a \sigma^a \gamma_5 \psi \text{ and } \delta\overline{\psi} = i\overline{\psi}\alpha^a \sigma^a \gamma_5$$

Consider,

$$\mathscr{O} = P^b(0) = \overline{\psi}(0)\sigma^b \gamma_5 \psi(0)$$

The Ward identity:

$$\langle \delta\mathscr{O} \rangle = \langle \mathscr{O} \delta S \rangle \Rightarrow$$

$$\langle \overline{\psi}\psi \rangle = m_q \int d^4x \langle P(x)P(0) \rangle$$

- In lattice one can get similar Ward identity where m is the PCAC quark mass (i.e. Goldstone bosons are massless in the $m \to 0$ limit) different from bare quark mass.
- By measuring the PCAC quark mass one can get the chiral condensate using the Ward identity.

Now,

$$\begin{aligned} & \int d^3x \langle P(x)P(0)\rangle \\ \approx \quad & \lim_{T\to\infty} \frac{1}{2m_\pi} |\langle 0|P(0)|1\rangle|^2 \{e^{-m_\pi|t|} + e^{-m_\pi|T-t|}\} = C_{PP} e^{-m_\pi|t|} \\ & m_\pi = E_1 - E_0, \quad C_{PP} = \frac{1}{2m_\pi} |\langle 0|P(0)|1\rangle|^2 \end{aligned}$$

Then,

$$\langle\overline{\psi}\psi\rangle = m_q \int d^4x \langle P(x)P(0)\rangle = m_q \int_{-\infty}^{\infty} dt C_{PP} \exp(-m_\pi |t|) = \frac{2m_q C_{PP}}{m_\pi} \quad (1$$

From axial Ward identity we have,

$$\partial_\mu A_\mu(x) = 2m_q P(x)$$

Taking the matrix element between the vacuum and physical pion state and then integrating over the volume,

$$m_\pi \langle 0 \mid A_4(0) \mid \pi\rangle = 2m_q \langle 0 \mid P(0) \mid \pi\rangle$$

which leads to

$$\begin{aligned} m_q &= \frac{m_\pi}{2}\sqrt{\frac{C^{AA}}{C^{PP}}} \\ \Rightarrow \langle\overline{\psi}\psi\rangle &= \frac{C_{AA} m_\pi}{2m_q} \end{aligned}$$

From the AA correlator

$$C(t) = \langle 0 \mid A_4(t) A_4(0) \mid 0 \rangle \xrightarrow{T\to\infty} C^{AA}\left[e^{-m_\pi t} + e^{-m_\pi (T-t)}\right]$$

$$C^{AA} = \frac{1}{2m_\pi} \mid \langle 0 \mid A_4(0) \mid \pi \rangle \mid^2 .$$

From PCAC,

$$\langle 0 \mid A_\mu(0) \mid \pi(p) \rangle = \sqrt{2} F_\pi p_\mu ,$$

$$\Rightarrow C_{AA} = F_\pi^2 m_\pi$$

$$\langle \overline{\psi}\psi \rangle = \frac{m_\pi^2 F_\pi^2}{2m_q} \quad \text{GMOR relation} \qquad (2)$$

Again by using PCAC one can get,

$$m_q^{AP} = \frac{m_\pi}{2} \frac{C^{AP}}{C^{PP}} \Rightarrow \langle \overline{\psi}\psi \rangle = C^{AP} . \qquad (3)$$

Renormalization Constants

From PCAC,

$$\partial_\mu A_\mu(x) = 2m_q P(x)$$
$$\Rightarrow z_A^{-1}\partial_\mu A_\mu^R(x) = z_P^{-1} z_{mA}^{-1} 2m_q^R P^R(x)$$
$$\Rightarrow \partial_\mu A_\mu^R(x) = z_A z_P^{-1} z_{mA}^{-1} 2m_q^R P^R(x)$$

$$\Rightarrow z_A = z_P z_{mA}$$

$$\begin{aligned}
\langle\overline{\Psi}\Psi\rangle &= \frac{2m_q C_{PP}}{m_\pi} = z_{mA}^{-1} z_P^{-2} \frac{2m_q^R C_{PP}^R}{m_\pi} = z_A^{-1} z_P^{-1} \langle\overline{\Psi}\Psi\rangle_R \\
\Rightarrow \langle\overline{\Psi}\Psi\rangle_R &= z_P z_A \langle\overline{\Psi}\Psi\rangle \\
&= z_P \langle\overline{\Psi}\Psi\rangle \quad \textit{In continuum}, \quad z_A = 1
\end{aligned}$$

R.C. for GMOR relation:

$$\langle\overline{\Psi}\Psi\rangle = \frac{m_\pi^2 F_\pi^2}{2m_q} = \frac{m_\pi C_{AA}}{2m_q} = z_A^{-2} z_{mA} \frac{m_\pi C_{AA}^R}{2m_q^R} = z_A^{-2} z_{mA} \langle\overline{\Psi}\Psi\rangle_R$$

$$\Rightarrow \langle\overline{\Psi}\Psi\rangle_R = z_A^2 z_{mA}^{-1} \langle\overline{\Psi}\Psi\rangle = z_A z_P \langle\overline{\Psi}\Psi\rangle$$
$$= z_P \langle\overline{\Psi}\Psi\rangle \quad \textit{In continuum}, \quad z_A = 1$$

Again,

$$\langle\overline{\Psi}\Psi\rangle = C^{AP}$$
$$\Rightarrow \langle\overline{\Psi}\Psi\rangle_R = z_A z_P \langle\overline{\Psi}\Psi\rangle$$

Table: Renormalization Constants for $\beta = 6/g^2 = 5.6$

R.C.	P.T.	B.P.T.
z_A	0.919	0.860
z_P	0.796	0.644
z_m	1.134	1.260

Table: Renormalization Constants for $\beta = 6/g^2 = 5.8$

R.C.	P.T.	B.P.T.
z_A	0.922	0.869
z_P	0.803	0.668
z_m	1.129	1.238

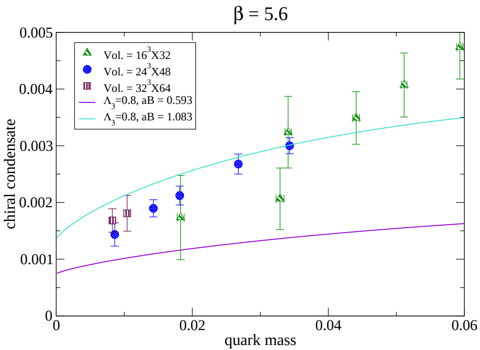

Figure: Renormalized Chiral condensate (using *GMOR*) vs renormalized quark mass in different lattice volumes and plots from CHPT formula in NLO for $aF_0 = .036$.

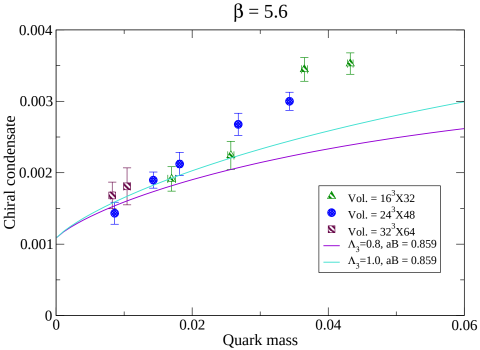

Figure: Renormalized Chiral condensate (using C_{PP}) vs renormalized quark mass in different lattice volumes and plots from CHPT formula in NLO for $aF_0 = .036$.

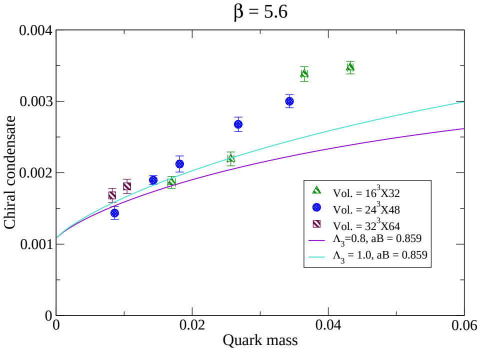

Figure: Renormalized Chiral condensate (using C_{AP}) vs renormalized quark mass in different lattice volumes and plots from CHPT formula in NLO for $aF_0 = .036$.

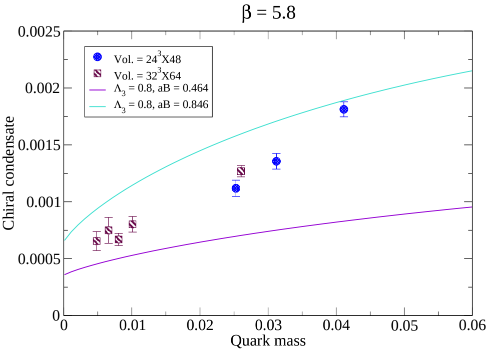

Figure: Renormalized Chiral condensate (using *GMOR*) vs renormalized quark mass in different lattice volumes and plots from CHPT formula in NLO for $aF_0 = .028$.

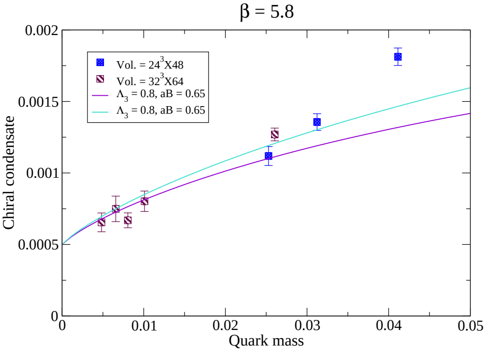

Figure: Renormalized Chiral condensate (using C_{PP}) vs renormalized quark mass in different lattice volumes and plots from CHPT formula in NLO for $aF_0 = .028$.

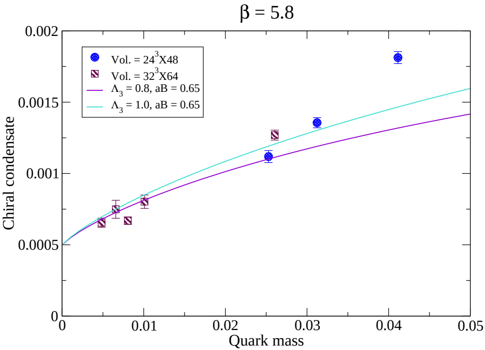

Figure: Renormalized Chiral condensate (using C_{AP}) vs renormalized quark mass in different lattice volumes and plots from CHPT formula in NLO for $aF_0 = .028$.

Summary

- Chiral condensate is difficult to calculate with Wilson fermion because of additive renormalization, but one can bypass it by using a Ward identity.
- Chiral condensate in chiral limit is supposed to vanish in finite volume. So, to calculate it one needs to take infinite volume limit followed by chiral limit.
- We need to calculate chiral condensate in larger volume and lower quark mass and incorporate non perturbative values of renormalization constants to pinpoint chiral condensate in chiral limit; but so far our result is reasonably well as far as CHPT is concerned in nonzero quark mass.

THANK YOU