Lattice QCD with Wilson Fermion

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Introduction

Here we present a detailed study of pion and rho mass , decay constants, and quark mass in Lattice QCD with two flavors of dynamical quark.

In LQCD, there are diferent formulations like Wilson, Staggered, Domain-Wall, Overlap, Twisted mass fermions etc. but each of them has advantages and disadvantages.

We use Wilson gauge and fermion action.

- ✓ Conceptually easy.
- ✓ Preserves discrete symmetries of continuum action.
  - × Explicit violation of chiral symmetry.
  - × Exceptional configuration.

The philosophy adopted here is to keep things simple at the fundamental level and stick to Wilsons formulation.

In future, it may be improved by different ways like using $\mathcal{O}(a)$ improvement, smearing the gauge links and the operators etc.

Wilson Fermion Action, parameters,...

Wilson Dirac operator:

$$D_W = \frac{1}{2}\{\gamma_\mu(\nabla_\mu^\dagger + \nabla_\mu) - \underbrace{ar\nabla_\mu^\dagger\nabla_\mu\}}_{\text{Wilson term.}} + m_0$$

This is an irrelevant term, goes to zero in the continuum.

The action is : $S = S_F + S_W + S_G$

The gauge part of the action is: $S_G = \beta \sum_P (1 - \frac{1}{3} ReTrU_P)$

The Wilson Dirac part of the action: $S_F + S_W = \sum_{x,y} \bar{\psi}_x M(U)_{x,y} \psi_y$

where $M(U)_{x,y} = \delta_{x,y} - \kappa[(r - \gamma_\mu)U_{x\mu} + (r + \gamma_\mu)U^\dagger_{x-\mu,\mu}]$

Parameters of the theory: $\beta = \frac{6}{g^2}$ and $\kappa = \frac{1}{2\hat{m}+8}$, where $\hat{m} = \frac{1}{2}[\frac{1}{\kappa} - \frac{1}{\kappa_c}]$

Chiral limit for free Wilson fermion: $\kappa_c^{\beta=\infty} = 0.125$, $\kappa_c^{\beta<\infty} > 0.125$

Critical slowing down and the cost of the algorithm forces us to work with finite β.

Symmetries of QCD

Consider $\mathcal{L}_{\mathcal{QCD}}$ for two degenerate flavor in chiral limit $[m = 0]$.

$\mathcal{L}_{\mathcal{QCD}} = \sum \bar{q}(\gamma_\mu \partial_\mu + ig\gamma_\mu A^a_\mu T^a)q$ for $q = (u \quad d)^T$

This Lagrangian has the following global symmetry.

$U(1)_V$: $q(x) \to e^{i\alpha} q(x)$, $\partial_\mu V_\mu = 0 \to$ Baryon number conservation.

$U(1)_A$: $q(x) \to e^{i\beta\gamma_5} q(x)$, $\partial_\mu A_\mu = \frac{g^2}{8\pi^2} Tr(G_{\mu\nu}\tilde{G}_{\mu\nu})$

$SU(2)_V$: $q(x) \to e^{i\alpha T^a} q(x)$, $\partial_\mu V^a_\mu = 0$

$SU(2)_A$: $q(x) \to e^{i\beta\gamma_5 T^a} q(x)$, $\partial_\mu A^a_\mu = 0$

Chiral symmetry group
$SU(2)_L \times SU(2)_R$.

$SU(2)_L \times SU(2)_R$ symmetry is spontaneously broken and reduces to only $SU(2)_V$ symmetry. $\Rightarrow$ appearence of three massless pseudoscalar bosons i,e. pions in the zero quark mass limit.

Approximate conservation of axial vector current is given by the PCAC hypothesis.

Chiral Symmetry on the Lattice

One of our main problem on the lattice is the question of chiral symmetry.

The problem is how to have right mass spectrum on the lattice and preserve chiral symmetry

$\Longleftarrow$ Impossible to preserve chiral symmetry on lattice while keeping also locality.(Nielsen Ninomiya Theorem.)

The Wilson term breaks the chiral symmetry in the kinetic part of the action even in the chiral limit.

$U(1)_V$ is still a symmetry but $U(1)_A$ is not.

$< a^{-1}\nabla^b_\mu A_\mu(x) >= 2m < \bar{\psi}(x)\gamma_5\psi(x) > + < X_x >,$ X_x depends on the Wilson term.

$\rightarrow$ In the continuum limit, it can be shown that X_x term reproduces the correct axial anomaly.

In continuum, PCAC tells that the divergence of non-singlet axial current vanishes in the chiral limit.

$$< \partial^\mu A^a_\mu(x) >= 0$$

Due to presence of Wilson term it appears on the lattice as

$$< \alpha|\nabla^\mu A^a_\mu|\beta >=< \alpha|[\bar{\psi}\{\tfrac{1}{2}T^a, m_0\}\gamma_5\psi + X^a]|\beta >$$

Redefinining A^a_μ, we get the PCAC relation in the continuum limit

$$Z_A < \alpha|\partial_\mu A^a_\mu|\beta >=< \alpha|\bar{\psi}\{\tfrac{1}{2}T^a, m_0 - \bar{m}\}\gamma_5\psi|\beta >$$

$$\bar{m} = \bar{m}(m_0, r, g_0).$$

The current A^a_μ is conserved for a critical value m_{cr} of m_0, such that

$$m_{cr} - \bar{m}(m_{cr}, r, g_0) = 0$$

m_{cr} is called the critical mass. Hence, the quark mass gets also additively renormalized.

$m_R = Z_m m_q$ where $m_q = m_0 - m_{cr}$, m_{cr} gives the true chiral limit of the Wilson fermion.

Exceptional Configurations

- Euclidean Dirac operator is anti-hermitian.
  - ✓ eigenvalues are purely imaginary or zero.
  - ✓ Fermion propagators have singularity only for vanishing mass parameter.
- Wilson term is hermitian.

  Properties of eigenvalues of Wilson-Dirac operator D_W:
  - ✓ appear in complex conjugate pairs (λ, λ^*).
  - ✓ obey reflection symmetry $(\lambda, -\lambda)$

Accidental zero eigenvalues of the fermion matrix are possible. Zero or near zero mode may occur for individual configuration at κ close to κ_{cr}.

⇒ Exceptional configurations: convergence problem of fermion matrix inversion.

Require large volume to be safe of accidental zero modes.

Hadron Mass calculation

A major goal (and a major virtue) of lattice QCD

→ Computation of hadron masses from first principle.

← evaluate Euclidean correltion functions.

Identify hadron operators $\mathcal{O}(\mathbf{x}, t)$, produce quantum numbers (spin,charge,parity) corresponding to the particle we are interested in.

For pion we choose PP and AP correlator where $P = \bar{q}_i \gamma_5 q_j$ and $A_4 = \bar{q}_i \gamma_0 \gamma_5 q_j$ and denotes the pseudoscalar density and fourth component of axial vector current.

For rho meson the correlator is VV where $V = \bar{q}\gamma_3 q$ or $T = \bar{q}\gamma_0\gamma_3 q$.

We are only interested in zero momentum, we can project $\mathcal{O}(t) = \sum_x \mathcal{O}(\mathbf{x}, t)$ on a $L^3 T$ lattice and find out the correlation function as a function of Euclidean time t.

For large t the lowest energy states dominates and we have

$$C_1(t) =< 0|\mathcal{O}^\dagger(t)\mathcal{O}(0)|0 > \xrightarrow{t\to\infty} C^{\mathcal{O}\mathcal{O}}[e^{-m_\pi t} + e^{-m_\pi(T-t)}]$$

$$C_2(t) =< 0|\mathcal{O}_1^\dagger(t)\mathcal{O}_2(0)|0 > \xrightarrow{t\to\infty} C^{\mathcal{O}_1\mathcal{O}_2}[e^{-m_\pi t} - e^{-m_\pi(T-t)}]$$

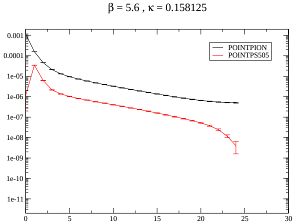
where the coefficients are given by

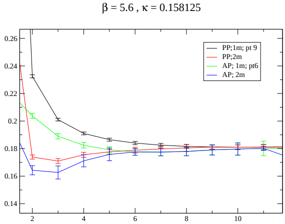
$$C^{\mathcal{O}\mathcal{O}} = \frac{1}{2m_\pi}| < 0|\mathcal{O}(0)|\pi > |^2 \quad \text{and} \quad C^{\mathcal{O}_1\mathcal{O}_2} = \frac{1}{2m_\pi} < 0|\mathcal{O}_1^\dagger(t)|\pi >< \pi|\mathcal{O}_2(0)|0 >$$

Cont...

We choose a range of $(t_{min} - t_{max})$ to determine the mass from the exponential fall.

Different types of exponential fits are done, like 1m-fit ($\sim C_1 e^{m_\pi t}$), 2m-fit ($\sim C_1 e^{m_\pi t} + C_2 e^{3 m_\pi t}$).


β = 5.6 , κ = 0.158125


β = 5.6 , κ = 0.158125

Quark mass and Decay Constants

The pion decay constant F_π and the quark mass m_q are defined in the continuum by the axial Ward identity.

$$< 0|A_\mu(0)|\pi(p) >= \sqrt{2}F_\pi p_\mu$$

$$\partial_\mu A_\mu(x) = 2m_q P(x)$$

From the PP and AP propagators

$$C^{PP} = \frac{1}{2m_\pi}| < 0|P(0)|\pi > |^2$$

$$C^{AP} = \frac{1}{2m_\pi} < 0|A_4(0)|\pi >< \pi|P^\dagger(0)|0 > |$$

which lead to: $F_\pi^{AP} = \frac{2\kappa C^{AP}}{\sqrt{m_\pi C^{PP}}}$ and $m_q^{AP} = \frac{m_\pi C^{AP}}{2C^{PP}}$

ρ decay constant is defined as

$$< 0|V_\mu(0)|\rho >= \epsilon_\mu \sqrt{2}F_\rho^V m_\rho$$

Here, ϵ_μ is the polarization vector of rho.

F_ρ^V is dimensionful and defined as

$\frac{F_\rho^V}{m_\rho} = 2\kappa\sqrt{\frac{C^{VV}}{m_\rho^3}}$, where $C^{VV} = \frac{1}{2m_\rho}| < 0|V_3(0)|\rho > |^2$

Systematic Limitations

In order to recover results that are appropriate to the original continuum lagrangian, there are three types of limitations:

- Finite size effects due to finite L
- Scaling violation due to finite a
- Large quark masses

Available range of a, L, m must be such that the result can be extrapolated to $L \to \infty$, $a \to 0$ and $m \to 0$.

Chiral Extrapolation

The cost of LQCD calculation forces the calculation to be done at unphysical large quark masses.

The results are extrapolated to physical quark masses using Chiral perturbation theory.

Chiral perturbation theory is applicable only when the bare quark masses are sufficiently low.

Simulation Details

Unimproved Wilson gauge and dynamical fermions

2 flavours of degenerate sea quarks

- $\beta = 5.6$

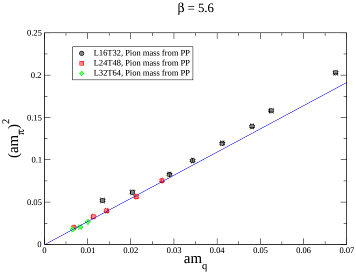
16^3**x32**	24^3**x48**	32^3**x64**
κ=0.156		
0.1565		
0.15675		
0.157		
0.15725		
0.1575	0.1575	
0.15775	0.15775	
0.158	0.158	
	0.158125	
		0.15815
	0.15825	0.15825
		0.1583

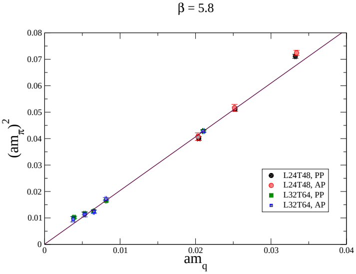
- $\beta = 5.8$

24^3**x48**	32^3**x64**
κ=0.1535	
0.1538	
0.154	0.154
	0.15455
	0.15462
	0.15470
	0.15475

The results shown in the following sections are based on preliminary data.

Results



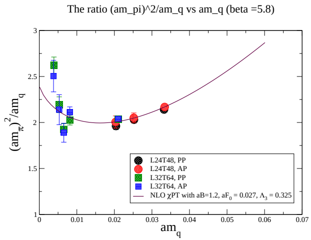


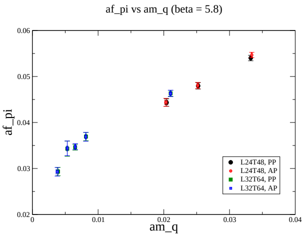
Since the chiral symmetry is approximate pions get mass. The data is consistent with $M_\pi^2 = 2Bm$.

Departure from $LO\chi PT$ shows up in the behaviour of the ratio $\frac{(am_\pi)^2}{(am_q)}$, as a function of am_q.

Chiral log appears at small quark mass region ($\sim$ 0-0.025).

Cont...


The ratio (am_pi)^2/am_q vs am_q (beta =5.8)

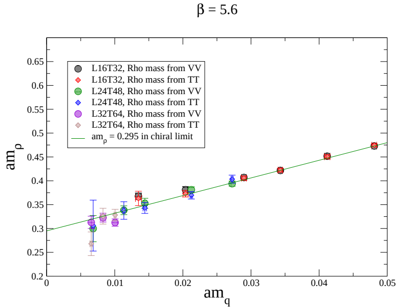

af_pi vs am_q (beta = 5.8)

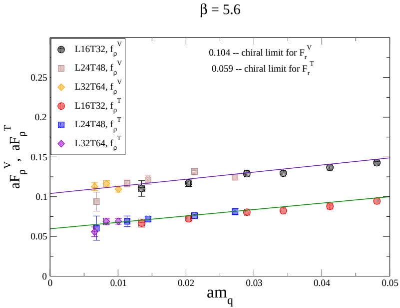
$$\frac{(am_\pi)^2}{(am_q)} = 2aB[1 - \frac{m_q B}{16\pi^2 F^2} ln \frac{\Lambda_3^2}{2m_q B}]$$

$$aF_\pi = aF[1 + \frac{m_q B}{8\pi^2 F^2} ln \frac{\Lambda_4^2}{2m_q B}]$$

Finite size effects are also important here.

Results for Rho mesons

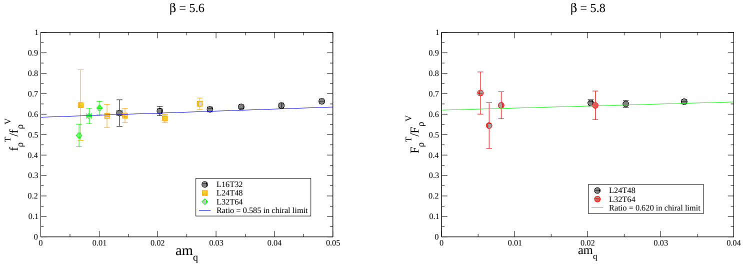




Chiral limit: $F_\rho^V \sim 258$mev.

$F_\rho^T \sim 146$mev.

Decay Ratio



Renormalization constants:

$Z_V = 0.759(0.775)$

$Z_T = 0.826(0.838)$

The ratio $\frac{F_\rho^T}{F_\rho^V} = 0.585(0.620)$ and the renormalized ratio is $0.636\ (0.670)$ for $\beta = 5.6(5.8)$.

Which is in perfect agreement with other recent determinations of this ratio and is very important for some B-decay rate calculations.

Summary

$\beta = 5.6; \ a = 0.0795 fm$

Smallest quark mass $\approx$ 20 MeV for $(2.5fm)^3$.

Pion mass ranging from 320 MeV to 550 MeV.

$M_\pi L = 4.25$ for smallest pion mass.

$\frac{m_\rho}{\sqrt{\sigma}} = 1.58.$ for chiral limit.

$\beta = 5.8; \ a = 0.0622 fm$

Smallest quark mass $\approx$ 15 MeV for $(2fm)^3$.

Pion mass ranging from 300 MeV to 850 MeV.

$M_\pi L = 3.12$ for smallest pion mass.

$\frac{m_\rho}{\sqrt{\sigma}} = 1.55.$ for chiral limit.
The approximate equality of $\frac{m_\rho}{\sqrt{\sigma}}$ for $\beta = 5.6$ & $\beta = 5.8$ may be an indication of absence of scaling violation.

Ongoing Simulations.

Data for $\beta = 5.7$ and larger lattice volume $(48^3 \times 64)$ are also ready to analyze.

Numerical calculation are carried out on a Cray XDI, supported by theory division, SINP under the DAE. This work is in part based on the MILC collaborations public lattice gauge theory code(http://physics.utah.edu/dtar/milc.html) and also on the DDHMC code developed by Luscher (http://luscher.web.cern.ch/luscher/DD-HMC/index.html) .

Thank You