Analytical Solutions of the DGLAP Equation for Non-Singlet Structure Function $F_2^{NS}(x,Q^2)$ At Small x

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- What is a non-singlet structure function ?

- The structure function of the proton can be decomposed into two parts:

- Flavour singlet and flavour non-singlet components:

$$F_2^P(x,Q^2) = \frac{5}{18} F_2^S(x,Q^2) + \frac{3}{18} F_2^{NS}(x,Q^2)$$

- The flavour non-singlet is basically a valance distribution of quarks which is given by:

$$F_2^{NS}(x,Q^2) = \frac{1}{3}\Big(\big(u(x,Q^2) + \bar{u}(x,Q^2)\big) - \big(d(x,Q^2) + \bar{d}(x,Q^2)\big)\Big)$$

$$= \frac{1}{3}\big(u_v(x,Q^2) - d_v(x,Q^2)\big)$$

- Evolution of the structure function with Q^2 is predicted by a set of integro-differential equations known as DGLAP equations:

- The DGLAP evolution equation for Non-singlet structure function $F_2^{NS}(x,Q^2)$ is

$$\frac{\partial F_2^{NS}(x,t)}{\partial t} = \frac{A_f}{t}\left[\{3+4\ln(1-x)\}F_2^{NS}(x,t) + 2\int_x^1 \frac{dz}{1-z}\left\{(1+z^2)F_2^{NS}\left(\frac{x}{z},t\right) - 2F_2^{NS}(x,t)\right\}\right]$$

Here, $t = \log\frac{Q^2}{\Lambda^2}$, $A_f = \frac{4}{3\beta_0}$, $\beta_0 = \frac{33-2n_f}{3}$, n_f being the number of quark flavours and Λ^2 is the QCD cut-off parameter.

- Now we introduce a variable $u = 1 - z$ and expand the argument $\frac{x}{z}$ in $F_2^{NS}\left(\frac{x}{z}, t\right)$ as a series: viz.

$$\frac{x}{z} = \frac{x}{1-u} = x\sum_{k=0}^{\infty} u^k = x + x\sum_{k=1}^{\infty} u^k$$

- Using the above equation we expand $F_2^{NS}\left(\frac{x}{z}, t\right)$ in a Taylor series as:

$$F_2^{NS}\left(\frac{x}{z}, t\right) = F_2^{NS}(x,t) + x\sum_{k=1}^{\infty} u^k \frac{\partial F_2^{NS}(x,t)}{\partial x} + \frac{\left(\sum_{k=1}^{\infty} u^k\right)^2}{2!}\frac{\partial^2 F_2^{NS}(x,t)}{\partial x^2} + \cdots.$$

- The series is convergent and so at low x we can approximate as:

$$F_2^{NS}\left(\frac{x}{z}, t\right) \approx F_2^{NS}(x,t) + x\sum_{k=1}^{\infty} u^k \frac{\partial F_2^{NS}(x,t)}{\partial x}$$

- The DGLAP equation now becomes:

$$\frac{\partial F_2^{NS}(x,t)}{\partial t} = \frac{A_f}{t}[\{3 + 4\ln(1-x)\}F_2^{NS}(x,t) + 2\int_x^1 \frac{dz}{1-z}(z^2-1)F_2^{NS}(x,t)$$
$$+ 2\int_x^1 \frac{dz}{1-z}(1+Z^2)\left(x\sum_{k=1}^{\infty} u^k\right)\frac{\partial F_2^{NS}(x,t)}{\partial x}$$

- We can write this Equation as,

$$\frac{\partial F_2^{NS}(x,t)}{\partial t} - \frac{8A_f}{3}\frac{x}{t}\frac{\partial F_2^{NS}(x,t)}{\partial x} = \frac{A_f\{4\ln(1-x) + 2x\}}{t}F_2^{NS}(x,t)$$

Which is a partial differential equation (PDF) in two variables x and t.

- We solve this equation by two methods:

  (1) Lagrange's Auxiliary Method. (LAM).
  (2) Method of characteristics. (MOC).

- In Lagrange's method the solution is obtained by solving the following system of auxiliary equations.

$$\frac{dx}{P(x)} = \frac{dt}{Q(t)} = \frac{dF_2^{NS}(x,t)}{R(x,t,F_2^{NS}}$$

Where $P(x) = -\frac{8A_f}{3}x$, $Q(t) = t$, $R\left(x,t,F_2^{NS}\right) = R'(x)F_2^{NS}(x,t)$.

With $R'(x) = A_f[4\ln(1-x) + 2x]$.

The general solution is given by $f(u,v) = 0$, where u , v are two independent solutions of the auxiliary system.

• Lagrange's Auxiliary Method leads us to a solution of the above equation for non-singlet structure function as,

$$F_2^{NS}(x,t) = \left(\frac{t}{t_\circ}\right)\frac{X^{NS}(1) - X^{NS}(x)}{\left(\frac{t}{t_\circ}\right)X^{NS}(1) - X^{NS}\ (x)}F_2^{NS}(x,t_\circ) \qquad -(A)$$

Here,

$$X^{NS}(x) = exp\left[-\int \frac{dx}{P(x)}\right] = exp\left[\frac{75}{32}\log\left(\frac{1}{x}\right)\right]$$

- By the method of characteristics the partial differential equation can be converted into an ordinary differential equation along the characteristic curves.

- The characteristics curve is given by the solution of the differential equation,

$$\frac{dx(t)}{t} = -\frac{8A_f}{3}\frac{x}{t}$$

- The ordinary differential equation is

$$\frac{dF_2^{NS}(x(t),t)}{dt} = C^{NS}(x(t),t)F_2^{NS}(x(t),t),$$

$$Where \quad C^{NS}(x(t),t) = \frac{A_f\{4\ln(1-x(t)) + 2x(t)\}}{t}.$$

- Integrating along the characteristics curve, the solution by this method is given as,

$$F_2^{NS}(x,t) = F_2^{NS}(\tau)\exp\left[\frac{3}{4A_f}x\left\{\left\{\left(\frac{t}{t_\circ}\right)^{\frac{8A_f}{3}} - 1\right\} - \frac{3}{2A_f}\left\{\left\{\sum_{k=1}^{\infty}\frac{x^k}{k^2}\left\{\left(\frac{t}{t_\circ}\right)^{\frac{8A_f}{3}} - 1\right\}\right\}\right\}\right]\quad -(B)$$

Where, $\tau = x\left(\frac{t}{t_\circ}\right)^{-\frac{8A_f}{3}}$, is the point at which characteristic curve cuts the initial curve.

COMPARISION OF THE TWO SOLUTIONS

- We compare these two results for F_2^{NS} obtained by two different methods with the data of F_2^{NS} from two experiments: CERN-WA25 (1991) and EMC Collaboration (1987).

- First, we plot the solutions as a function of x at some different fixed Q^2. For the evolution we take the input from MRST2001.

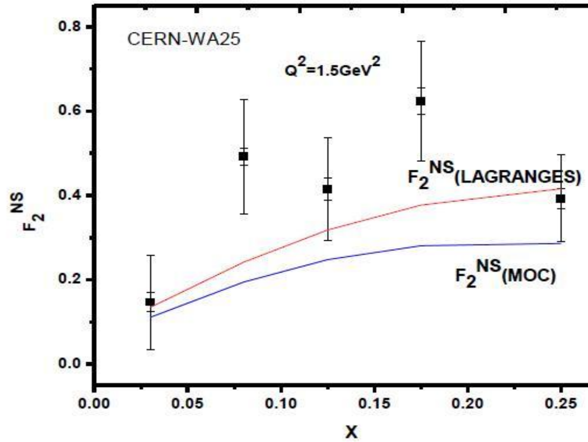
- Second, we plot the solutions as a function of Q^2 at different fixed x.

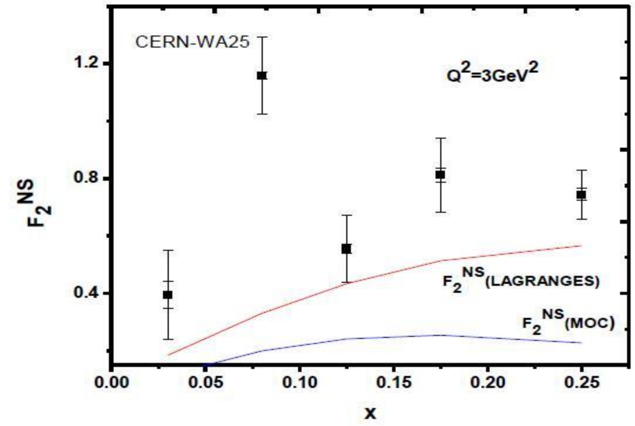
- In order to test that the goodness -of-fit among the two results with the experimental data of F_2^{NS}, we do the x^2 testing for both the results.

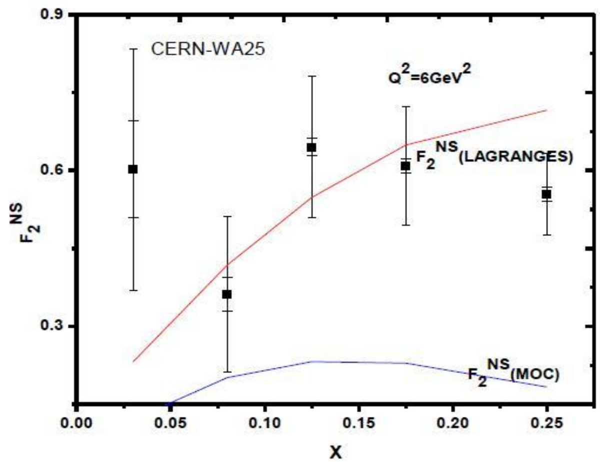
- We obtain the information about the non-singlet structure function from the data of proton and neutron structure function using the relation,

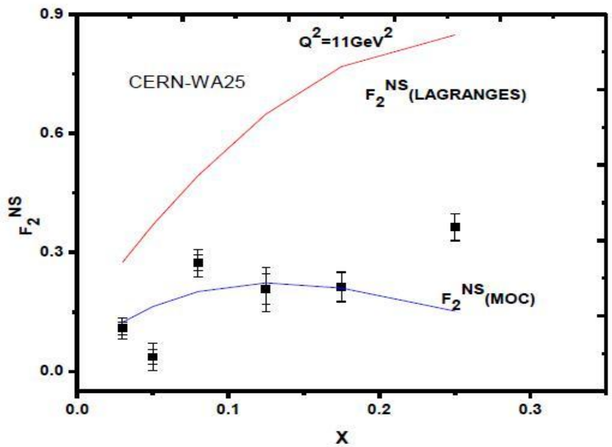
$$F_2^{NS} = 3(F_2^P - F_2^n)$$

F_2^{NS} as a function of x at different fixed Q^2

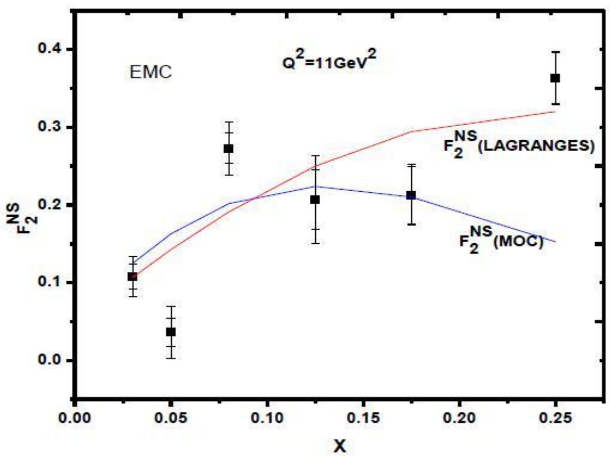


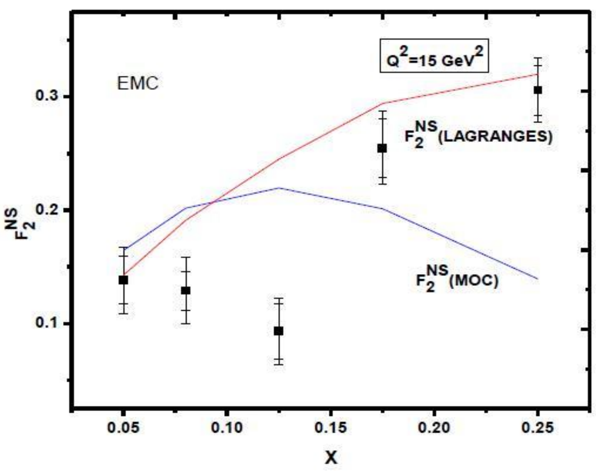




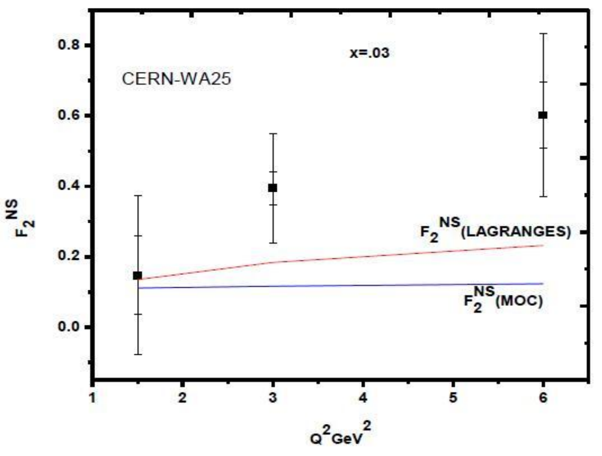


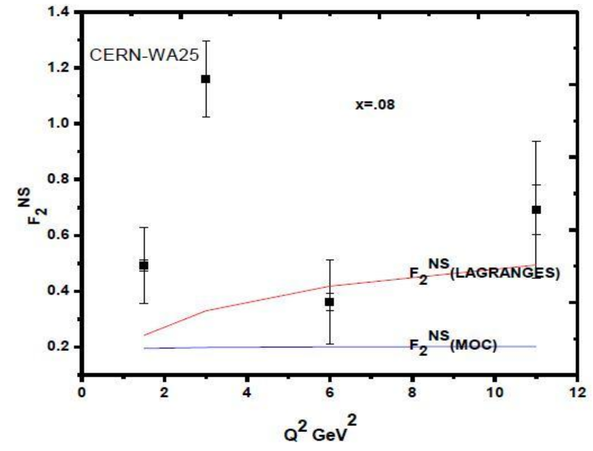
F_2^{NS} as a function of x at different fixed Q^2

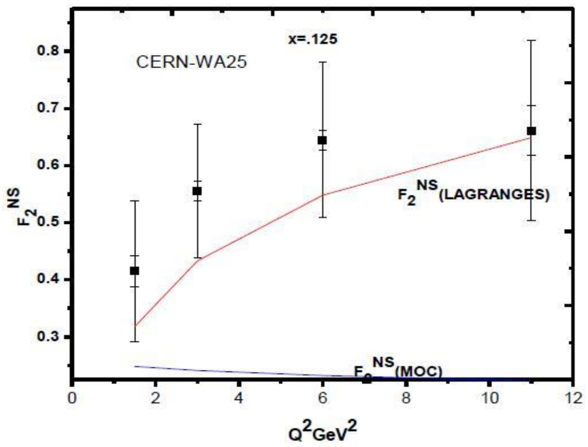


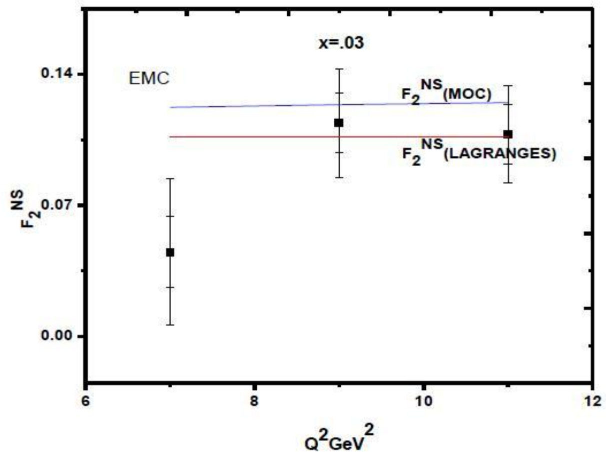


F_2^{NS} as a function of Q^2 at different fixed x.

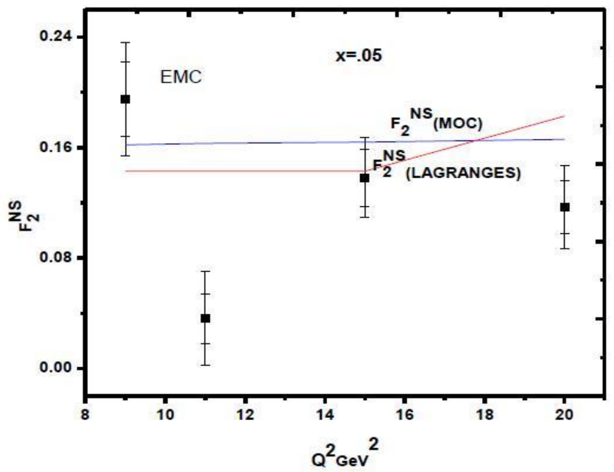


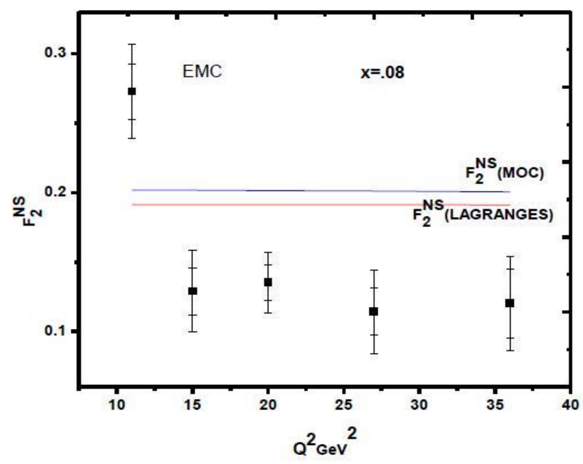


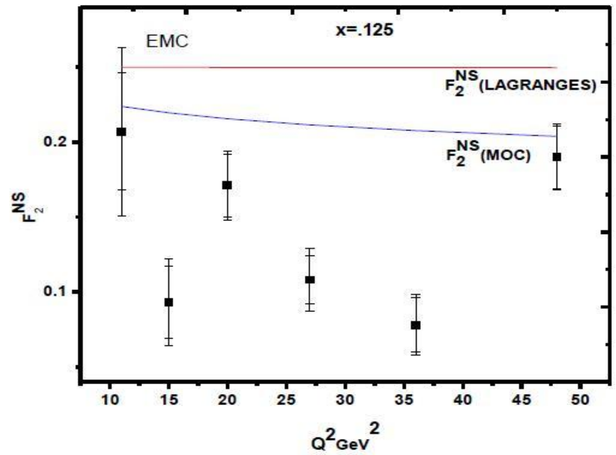




F_2^{NS} as a function of Q^2 at different fixed x.







The Chi square, $(\frac{\chi^2}{N})$ values calculated for the two solutions (A) and (B) are listed below:

Chi square values of the methods	CERN-WA25. $Q^2<10GeV^2$	EMC Collaboration. $Q^2>11GeV^2$
Method of Characteristics	0.338	0.243
Lagrange's Method	0.175	0.958

We calculate Chi square value using the formula,

$$\chi^2 = \sum_{i \in data\ points} \left(\frac{X_{Th} - X_{Exp}}{\sigma^2}\right)^2$$

CONCLUSION

- From the first set of graphs at different fixed Q^2 values we can conclude that the solution by Lagrange's Auxiliary Method is much more appropriate in low x, low Q^2 region ($x<0.25$, $Q^2<6GeV^2$) compared to MOC.

- But at high Q^2 ($Q^2 \geq 11GeV^2$), low x ($x<0.25$) region, the solution by the Method of Characteristics is observed to be closer to the data.

- Again, comparing the two solutions at fixed x values for different Q^2 region, we see that at low x ($x<.125$), solutions by Lagrange's Method is observed to be closer to the experimental data at both low and high Q^2 (Q^2 up to 50 GeV^2).

- But as we move towards the higher x and higher Q^2 region ($x \geq .125$, $Q^2>10GeV^2$) the result by Method of Characteristics appears to be more applicable.

- Analysing the Chi square value, also we can conclude that at low x region the Lagrange's Auxiliary Method gives us the better result for non-singlet structure function when Q^2 is also low ($Q^2<6GeV^2$).But in case of high Q^2 region ($Q^2>10GeV^2$) Method of Characteristics gives us a better result of the DGLAP equation for the non-singlet structure function.

THANK YOU